Analytical Investigation of Plate Flutter in Supersonic Gas Flow

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There are many investigations which are devoted to the plate flutter problem. The solution is reduced to calculation of system eigenfrequencies, their position on the complex plane indicates existence or absence of the flutter. For gas pressure one uses the piston theory or its modifications valid for high enough Mach numbers, and solves the eigenvalue problem numerically.

In the present paper flutter of the plate having form of wide strip is investigated. For eigenvalue problem the asymptotic method for extended areas, or the theory of global instability [1; 2, §65] is used. For gas pressure is used asymptotically exact expression at $L \to \infty$ for full range of Mach numbers $M > 1$. Two qualitatively different types of instability are obtained. First, single oscillation form flutter, which is result of negative aerodynamic damping. Second, coupled-type flutter, which is result of interaction of two oscillation forms. Stability criteria and the frequencies corresponding to maximum amplification are derived for both types. Mechanism of transition of eigenfunctions to instability has a simple physical sense and is described in details below.

1. Statement of the problem

In plane statement one considers a linear stability of elastic strained plate having form of strip which is streamlined from one side by homogeneous supersonic gas flow and balanced from another side by constant pressure. Gas flow vector is parallel to the plate plane. One considers the gas as inviscid and perfect, the flow assumed to be adiabatic, plate obeyed the classic equation of thin plate bending.

Assume all variables dimensionless and introduce the coordinate system connected with the plate as shown in fig. 1. The plate occupies the region $|x| \le L/2$, $z = 0$, here L is width of the plate; at $|x| > L/2$ surface $z = 0$ is considered as absolutely rigid.

Apply to system small perturbance described by gas velocity potential φ and plate bending w.

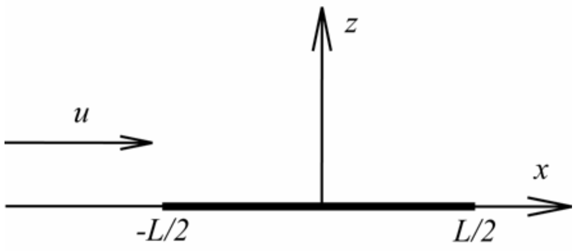

Fig. 1

In the region $z > 0$ gas potential φ satisfies the wave equation

$$\left(\frac{\partial}{\partial t} + M\frac{\partial}{\partial x}\right)^2 \varphi - \frac{\partial^2 \varphi}{\partial x^2} - \frac{\partial^2 \varphi}{\partial z^2} = 0$$

and the damping condition at $z \to \infty$. At $z = 0$ potential obeys impenetrability condition

$$\frac{\partial \varphi}{\partial z} = \frac{\partial w}{\partial t} + M\frac{\partial w}{\partial x},\ |x| < \frac{L}{2};\quad \frac{\partial \varphi}{\partial z} = 0,\ |x| > \frac{L}{2}.$$

The bending w obeys the equation of motion of the plate

$$\frac{\partial^2 w}{\partial t^2} = \mu\left(\frac{\partial \varphi}{\partial t} + M\frac{\partial \varphi}{\partial x}\right) + M_w^2 \frac{\partial^2 w}{\partial x^2} - D\frac{\partial^4 w}{\partial x^4}$$

($|x| < L/2$), and the boundary conditions at plate edges. Undimential parameters are as follows:

$$M = \frac{u}{a},\qquad M_w = \frac{\sqrt{\sigma/\rho_m}}{a},\qquad D = \frac{D_w}{a^2 \rho_m h^3},$$

$$L = \frac{L_w}{h},\qquad \mu = \frac{\rho}{\rho_m}$$

Here u and ρ are velocity and density of the gas, a is its sound speed, σ is strain stress at middle surface of the plate, ρ_m, L_w and h are material density, width and thickness of the plate, $D_w = \dfrac{Eh^3}{12(1-\nu^2)}$ is its bending stiffness. Parameter M_w is the ratio of propagation speed of long bending plate waves to the speed of sound in the gas. Assume that $M > 1$, $\mu << 1$, $L >> 1$.

Supposing all functions depending on time as $e^{-i\omega t}$ and substituting this dependence in the system of equations, one obtains the eigenvalue problem. Flutter criterion is existence of at least one eigenvalue having positive imaginary part.

2. Global instability

In [1; 2, §65] general method for solving the eigenvalue problem is developed and on its basis sufficient instability conditions for homogeneous states of one-dimensional extended systems of general form are formulated. Two types of instability were discovered: one-side instability which depends on boundary conditions at one of system boundaries, and global instability which does not depend on the boundary conditions. Plate boundary conditions usually used are pinning ($w = \partial w/\partial x = 0$) and clamping ($w = \partial^2 w/\partial x^2 = 0$); they do not satisfy one-side instability condition of plane perturbances.

Global instability criterion is as follows. Let dispersion relation of unbounded (i.e. occupying full x axis) system is $F(k,\omega) = 0$, where k is wave number, ω is complex frequency of the perturbance. At sufficiently large $\operatorname{Im}\omega$ its solutions $k_j = k_j(\omega)$, which are numbered in imaginary part decrease, can be divided into two groups: $\operatorname{Im} k_j > 0$, $j=1,\dots,s$ and $\operatorname{Im} k_j < 0$, $j=(s+1),\dots,N$, the number of solutions in each group is equal to the number of boundary conditions at one of boundaries of finite system. Every solution defines a branch of multi-valued analytical function $k = k(\omega)$. When $\operatorname{Im}\omega$ decreases, imaginary parts of the first group solutions decreases, and ones of the second group increases, and for some ω the equality

$$\min_{1\le j\le s} \operatorname{Im} k_j = \operatorname{Im} k_m = \operatorname{Im} k_n = \max_{s+1\le j\le N} \operatorname{Im} k_j$$

will be satisfied. Set of such ω defines a curve Ω on the complex plane ω. Under sufficiently large extension of finite system, part of its eigenfrequencies spectrum lies near this curve, as more densely and closely to it, as more the system extension [1; 2, §65]. At that eigenfunctions corresponding to indicated eigenfrequencies under $L \to \infty$ and far from boundaries of the system have asymptotic form $\left(C_m e^{ik_m(\omega)x} + C_n e^{ik_n(\omega)x}\right) e^{-i\omega t}$. Here C_j are constants defined by boundary conditions and i is imaginary unit. It is sufficient for instability of the system that a part of curve Ω would lie in the region $\mathrm{Im}\,\omega > 0$; this is condition of global instability.

Dispersion relation for plane perturbances of unbounded plate which is streamlined from one side by gas has the form [3]:

$$(Dk^4 + M_w^2 k^2 - \omega^2) - \mu \frac{(\omega - Mk)^2}{\sqrt{k^2 - (\omega - Mk)^2}} = 0 \tag{1}$$

First term corresponds to contribution of elastic and inertial forces of the plate, the second one corresponds to contribution of aerodynamic forces applied to the plate. At $\mathrm{Im}\,\omega >> 1$ square root in the second term is chosen such that its real part is positive; this condition is the consequence of damping condition away from the plate. It picks out four solutions of (1) $k(\omega)$, analytically continued on the complex plane ω cut by line $\mathrm{Re}\,\omega = 0$, $\mathrm{Im}\,\omega < 0$. At large $|\omega|$ solutions are close to roots of dispersion relation of unbounded plate in vacuum ($\mu = 0$); to each group belong two roots.

Note that it follows from (1) that curve Ω is symmetric relative to imaginary axis ω. Thus we can consider only the right half-plane of the complex plane ω.

Results of our investigation are described below, analysis details can be found in [4].

3. Global instability of high-frequency perturbances

In the present section assume that μ is small parameter and ignore low-frequency perturbances on assuming that $\omega >> \mu$, $k >> \mu$. Then the second term in (1) is small in comparison with the first one. Since at $\mu = 0$ the curve Ω coincides with real axis, then at small $\mu \neq 0$ this curve consists of points $\omega + i\delta(\omega)$, $0 < \delta << 1$ $\omega \in \mathbf{R}$, $\omega > 0$. Expanding $k_j(\omega + i\delta, \mu)$ in Taylor formula in δ and in μ, one obtains from the condition $\mathrm{Im}\,k_2(\omega + i\delta, \mu) = \mathrm{Im}\,k_3(\omega + i\delta, \mu)$ dependence $\delta(\omega)$ (so the form of the curve Ω) and instability criterion.

Flutter criterion accurate for infinitesimal order of μ is the inequality

$$M > M_w + 1, \tag{2}$$

its physical sense is in the following.

First, consider the plate of infinite width. Suppose that harmonic wave $w(x,t) = e^{i(kx - \omega t)}$ runs along the plate with phase velocity $c = \omega / k = \sqrt{M_w^2 + Dk^2}$; $\omega \in \mathbf{R}$. The wave generates perturbance of pressure which in turn leads to correction in the wave number k. Spatial amplification or damping of the wave (under real frequency ω), which is defining by $\mathrm{Im}\,k(\omega)$, depends on the character of gas flow relative to the wave. If the flow is subsonic, that is $|M - c| < 1$, than the phase of pressure perturbance coincides with the phase of plate bending and cannot lead to damping or amplification of the wave. If the flow is supersonic than the phase of pressure is shifted by $\pi/2$ with respect to the phase of the wave, in this case spatial damping or amplification of the wave depends on the direction of gas flow with respect to the wave, and the direction of wave motion. If they coincide than the gas performs positive work and the wave is amplified, if they are counter than the gas performs negative work and the wave is damped. Thus the wave moving against the gas flow always experiences resistance from it and is damped, the wave moving along the gas flow is amplified if the gas moves faster than the wave ($M - c > 1$)

and is damped if the wave moves faster than the flow ($M - c < -1$).

Now consider the plate of large but finite width. Mechanism of generation of its global eigenfunctions which are built in accordance with [1; 2, §65] is of the following form. The wave corresponding to wave number k_2, which is running along the plate downstream the gas flow, reflects from its back edge and generates two waves running back. One is damped and corresponds to k_4, the second is close to harmonic wave and corresponds to k_3. If the plate is sufficiently wide then when these waves reach the forward edge, amplitude of damping wave is negligible and one may assume that only wave, corresponding to k_3, came back. Then it reflects from the forward edge and generates damping wave corresponding to k_1 (which is is negligible at the back edge by the same reason) and original wave, corresponding to k_2, but generally having different amplitude. As a consequence of described process of mutual conversion of both waves in reflections (two appending damping waves are essential only near plate edges), the global eigenfunction is generated. If amplification of the wave running downstream becomes more than damping of the wave running upstream, then after double reflection wave amplitude will be increased. On cyclic iterations of this process this leads to exponential increase of perturbance amplitude. Amplification of the wave running downstream takes place when $c < M - 1$, at that it becomes as larger as c (phase speed of the wave running along the plate) is closer to $M - 1$ (motion speed of back front of sonic waves in the gas). Under $c = M - 1$ these waves resonate, which leads to their maximum amplification. As $c(k) = \sqrt{M_w^2 + Dk^2}$, the criterion (2) which is accurate to small terms of order of μ corresponds to parameters values region for which there exist waves along the plate with phase speed $c = M - 1$, so that $k >> \mu$.

Using the fact that waves, which phase speed $c = M - 1$, grow the most quickly, from dispersion relation (1) one can find their frequency:

$$\omega_{\max} = (M-1)\sqrt{((M-1)^2 - M_w^2)/D}$$

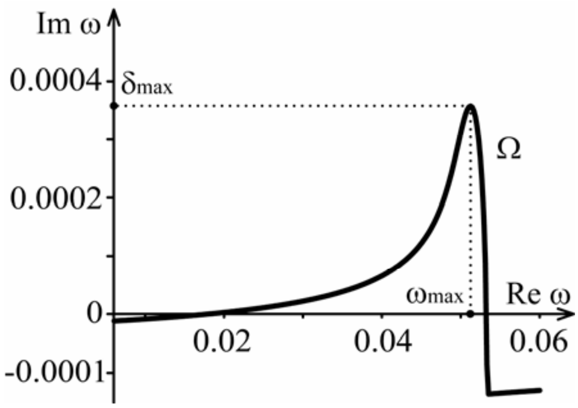

Fig. 2

In fig. 2 the part of the curve Ω is shown for parameters

M=1.5, M_w=0, D=23.9, μ=1.2·10^{-4} (3)

(steel plate streamlined by an air flow under normal conditions) from which, in particular, one can find maximum increment of oscillations $\delta_{\max} \approx 3.6 \cdot 10^{-4}$. All points of the curve Ω, which do not lie on the shown region of ω plane and satisfy condition $\omega >> \mu$, have negative imaginary part.

Consider now descrete spectrum of system eigenfrequencies.

In the absence of the gas eigenfrequencies ω_n lie on the real axis ω. In the presence of the gas eigenfrequencies, which are satisfy condition $\omega_n >> \mu$, in first approach under large L lie on curve Ω and have the form $\omega_n + i\delta(\omega_n)$.

As a consequence the following dependence of plate eigenfunctions growth rate on its width is generated. Under sufficiently small width L (on assumption that theory of global instability can be applied for considered L) $\omega_n >> \omega_{\max}$ for any n and state of the plate is stable because $\delta(\omega_n)<0$. In increasing L eigenfrequencies move in the direction of their real part decrease, and frequencies ω_1, ω_2 etc in series move through area $\delta(\omega)>0$. As the distance between neighbour frequencies which lie near $\omega_{\max}$ at $L\to\infty$ tends to zero, then beginning from some L the plate always will belong in high-frequency flutter region, and maximum growth rate among all high-frequency eigenfunctions $\delta_w = \max\limits_{n:\omega_n >> \mu} \delta(\omega_n)$ in increase of L will have the form of damping oscillations, asymptotically approaching to $\delta_{\max}$. For example, the dependence of δ_w on pinned plate width for parameters (3) is shown in fig. 3. To each successive local minimum δ_w when L increases, transition of the quickest growth to the successive oscillation eigenform corresponds.

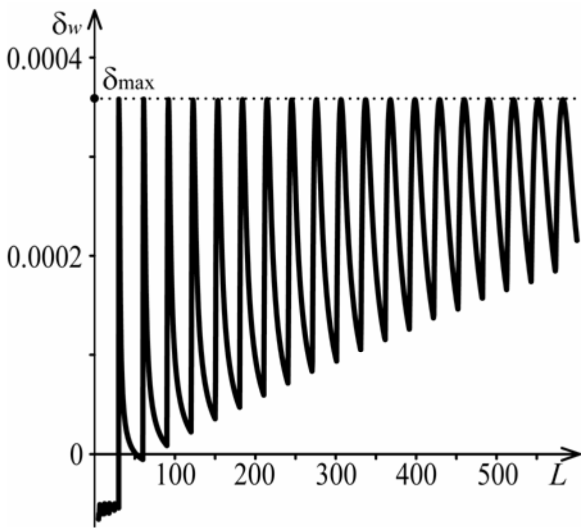

Fig. 3

Analogously one can trace the dependence of eigenfrequencies $\omega_n >> \mu$ on Mach number M. At that eigenfrequencies of plate oscillations in vacuum ω_n remain motionless and the curve Ω for increasing M moves to the right and deforms. Dependence $\delta_w(M)$ also has oscillation character, but maximum growth rate $\delta_{\max}$ changes with M alteration. For example, the dependence $\delta_{\max}(M)$ is shown in fig. 4 for parameters D=23.9, $\mu=1.2\cdot10^{-4}$.

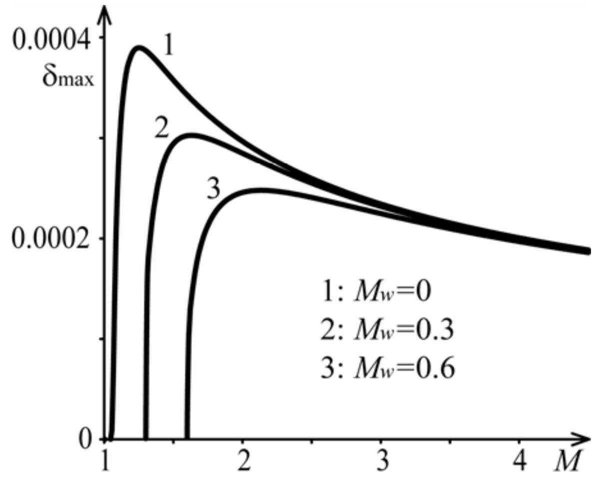

Fig. 4

4. Global instability of low-frequency perturbances

Consider now low-frequency perturbances when the second term in (1) is not small in comparison with the first one.

Let $M_w = 0$. It follows from previous section that at sufficiently large $|\omega|$ the curve Ω is defined by the condition $\operatorname{Im} k_1(\omega) > \operatorname{Im} k_2(\omega) = \operatorname{Im} k_3(\omega) > \operatorname{Im} k_4(\omega)$. Let us move along Ω in the direction of $\operatorname{Re}\omega$ decrease. Then this condition will be satisfied only up to some ω defined by the expression $\operatorname{Im} k_1(\omega) > \operatorname{Im} k_2(\omega) = \operatorname{Im} k_3(\omega) = \operatorname{Im} k_4(\omega)$; at this point the curve Ω has sharp turn. At following $\operatorname{Re}\omega$ decrease mutual position of branches k_3 and k_4 is changed, and points of the curve Ω are defined by the condition $\operatorname{Im} k_1(\omega) > \operatorname{Im} k_2(\omega) = \operatorname{Im} k_4(\omega) < \operatorname{Im} k_3(\omega)$. The curve ends at the branch point ω^* of roots $k_2(\omega)$ and $k_4(\omega)$, at which $\operatorname{Im}\omega^* > 0$. Thus, at $M_w = 0$ low-frequency perturbances are unstable. For example, the curve Ω for small ω region under parameters (3) is show in fig. 5.

Physical sense of mutual changing of the position of $\operatorname{Im} k_3$ and $\operatorname{Im} k_4$ at small $\operatorname{Re}\omega$ consists in the following. In the absence of the gas, for small real ω the wave, which corresponds to branch k_3, is harmonic, and the wave, which corresponds to k_4, is low-damping, these both waves propagate against the flow. Under gas influence the wave corresponding to k_3 also becomes damping, so that at small ω its damping has the same order as the damping of wave k_4. Change of inequality $\operatorname{Im} k_3 > \operatorname{Im} k_4$ to inequality $\operatorname{Im} k_3 < \operatorname{Im} k_4$ means that at small frequencies damping of wave k_3 under gas influence becomes stronger than the damping of wave k_4, and global eigenfunctions are generated by waves k_2 and k_4 instead of k_2 and k_3.

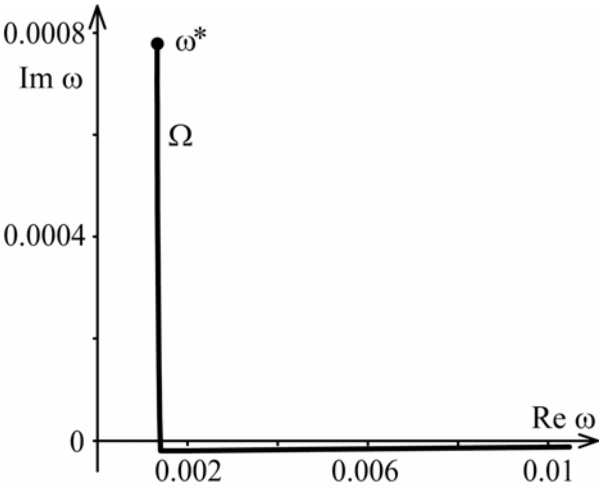

Fig. 5

Stability investigation of low-frequency perturbances is based on study of position of roots $k_2(\omega)$ and $k_4(\omega)$ branch point ω* on ω plane. On condition $k >> \omega$ (it can be shown that it is correct for the branch point) one can simplify the dispersion relation (1), neglecting ω in the second term and choosing certain branch of the square root:

$$(Dk^4 + M_w^2 k^2 - \omega^2) + i\mu \frac{M^2}{\sqrt{M^2 - 1}} k = 0 \quad (4)$$

Investigation of (4) leads to the following instability criterion:

$$M_w < \left(\frac{\sqrt{54}}{4} \right)^{1/3} \left(\mu \frac{M^2}{\sqrt{M^2 - 1}} \right)^{1/3} D^{1/6}.$$

The frequency, which corresponds to maximum of perturbances growth rate, has the form

$$\omega = A \left(\mu \frac{M^2}{\sqrt{M^2 - 1}} \right)^{2/3} D^{-1/6},$$

where A depends on parameters of the problem and lies in the range from 0.433 to 0.595. In the case of instability the curve Ω has the same qualitative form as shown in fig. 5. In the case of stability no mutual interchanging of k_3 and k_4 at small ω is present. At that the curve Ω is ended in the branch point of roots k_2 and k_3 lying in the under half-plane.

5. Discussions

Essentially the method of global instability which is used in solution of eigenvalue problem, leads to the following method of calculation of pressure acting on oscillating plate: oscillation is assumed to be a superposition of waves running along imaginary unbounded plate, and satisfying the same boundary conditions at edges as for the real plate; pressure is considered as the superposition of pressures acting on these unbounded waves. On using exact dependence of pressure on bending [5], one can show that under $\operatorname{Im} \omega > 0$ this method at $L \to \infty$ is asymptotically exact.

Suppose that for the pressure acting on unbounded running waves were used expression

$$p(x,t) = C_1 \frac{\partial w}{\partial x} + C_2 \frac{\partial w}{\partial t} \quad (5)$$

where C_j are functions of parameters of the problem, $C_2 \geq 0$ (in particular, such form has the piston theory and some other approaches at large Mach numbers). Then high-frequency perturbances would be stable. Thus, high-frequency flutter cannot be obtained in using of expressions having form (5). Note, that the theory discovered in [6], correctly describe high-frequency flutter and leads to the criterion which coincides with derived here.

Low-frequency flutter is described correctly by dependences of the form (5), for the reason that in its investigation one used approximate dispersion relation (4) which can be treated as exact dispersion relation derived in use of quasi-static approach having the form (5), where $C_1 = \mu \frac{M^2}{\sqrt{M^2-1}}$, $C_2 = 0$.

Now explain physical difference between high-frequency and low-frequency flutter. In [7,8] two types of plate flutter are described — single degree of freedom flutter and coupled-type flutter. The first one appears under action of negative aerodynamic damping of one of oscillation form, so that there is no interaction between forms. The second type of flutter is the consequence of two eigenforms interaction [5,9]. Prove that derived in the present paper high-frequency and low-frequency flutter are accordingly single degree of freedom flutter and coupled-type flutter. Really, it is easy to understand that the structure of high-frequency growing eigenfunction described in section 3 coincides with structure of eigenfunctions of the plate being in the vacuum. In other words, oscillations are in a single oscillation form. On the other hand, it follows from results of [5,9] derived in the approach of the piston theory that the plate having the form of strip of sufficiently large width is being in coupled-type flutter region. As high-frequency flutter cannot be discovered in approach of the piston theory, coupled-type flutter is low-frequency flutter.

The correspondence derived between both flutter types clarifies the fact that the single degree of freedom flutter was described only in works [7,8]. In overwhelming works one used dependences having form (5), which, as shown above, do not lead to existence of single degree of freedom flutter. But there where more exact dependences of pressure on bending are used, seemingly during numerical solution one used too small terms of series for discovery of high-frequency flutter. Note that in aircraft design criteria [10-12] for investigation of criterion of paneling flutter quasi-static and piston theory are used, and thus the possibility of single degree of freedom flutter is not excluded.

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