Fluid Dynamics, Vol. 40, No. 5, 2005, pp. 805–817. Translated from Izvestiya Rossiiskoi Academii Nauk, Mekhanika Zhidkosti i Gaza, No. 5, 2005, pp. 155–169. Original Russian Text Copyright © 2005 by Vedeneev.

Flutter of a Wide Strip Plate in a Supersonic Gas Flow

V. V. Vedeneev

Received December 22, 2004

Abstract — The stability of an elastic plate in the form of a wide strip in a supersonic inviscid gas flow is investigated in the linear approximation. An expression for the dependence of the pressure on the plate deflection, asymptotically exact for wide plates, is used. Two qualitatively different instability types are obtained: flutter with respect to a single oscillatory mode due to negative aerodynamic damping and flutter of a related type due to the interaction of oscillatory modes. For each type the stability criterion and the frequency at which the oscillation amplitude grows most intensely are found.

Keywords: plate flutter, panel flutter, plate stability, global instability, aerodynamic damping.

In [1–11], the problem of the flutter of an elastic plate in a supersonic gas flow was considered in various formulations. The solution reduced to finding the eigenvalues of a certain operator, and the presence or absence of flutter was determined by their position in the complex plane. For the gas pressure, as a rule [2–4, 7–9], the piston theory [3] or its modifications valid for fairly high Mach numbers were used and the eigenvalue problem was solved numerically.

In this study, the eigenvalue problem is solved using an asymptotic method for long domains, known as global instability theory [12; 13, § 65], while the gas pressure is represented by an expression asymptotically exact as the plate width $L \to \infty$ over the entire range of Mach numbers $\mathrm{M} > 1$. The conditions for which a wide strip plate in a supersonic gas flow has growing global eigenfunctions are found analytically. Their mechanism of origin and destabilization, which admits a physically transparent interpretation, is described.

1. FORMULATION OF THE PROBLEM

We will investigate in the linear approximation the stability of a stretched elastic plate in the form of an infinite strip exposed on one side to a homogeneous supersonic flow and balanced by a constant pressure on the other. The gas velocity vector is parallel to the plate plane and the angle between the flow direction and the plate edges is equal to $\pi/2 - \theta$ (Fig. 1), $0 \le \theta < \pi/2$. We assume that the gas is inviscid and perfect and the flow adiabatic; the plate obeys the classical bending equation for a thin plate.

We introduce a system of coordinates tied to the plate with the x axis lying in the plane of the plate perpendicular to the edges, the y axis parallel to the edges, and the z axis normal to the plate so that the coordinate system xyz is positively oriented. We will assume that in the undisturbed case the gas flows in the region $z > 0$ and the plate occupies the region $|x| \le L_w/2$, $z = 0$, where L_w is the plate width. At $|x| > L_w/2$ the surface $z = 0$ is assumed to be absolutely rigid.

In dimensionless variables the differential equations describing the evolution of small perturbations independent of the y coordinate can be written as follows:

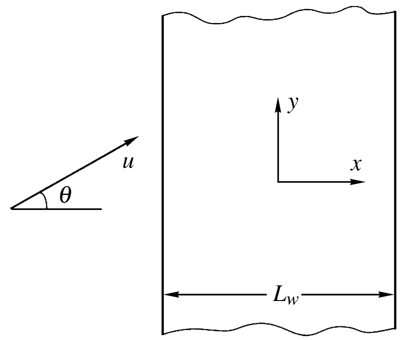

Fig. 1. Configuration of the system considered

$$\left(\frac{\partial}{\partial t} + \mathrm{M}\frac{\partial}{\partial x}\right)^2 \varphi - \frac{\partial^2 \varphi}{\partial x^2} - \frac{\partial^2 \varphi}{\partial z^2} = 0, \quad z > 0$$

$$\frac{\partial \varphi}{\partial z} = \frac{\partial w}{\partial t} + \mathrm{M}\frac{\partial w}{\partial x}, \quad z = 0, \quad |x| < \frac{L}{2}, \qquad \frac{\partial \varphi}{\partial z} = 0, \quad z = 0, \quad |x| > \frac{L}{2}$$

$$\frac{\partial \varphi}{\partial q} = 0, \quad z \to +\infty, \quad q = x,\ z,\ t \tag{1.1}$$

$$\frac{\partial^2 w}{\partial t^2} = \mu \left(\frac{\partial \varphi}{\partial t} + \mathrm{M}\frac{\partial \varphi}{\partial x}\right) + \mathrm{M}_w^2 \frac{\partial^2 w}{\partial x^2} - D\frac{\partial^4 w}{\partial x^4}, \quad z = 0, \quad |x| < \frac{L}{2}$$

$$F_1 w = 0, \quad z = 0, \quad x = -\frac{L}{2}, \qquad F_2 w = 0, \quad z = 0, \quad x = \frac{L}{2}$$

Here, φ and w are the dimensionless gas perturbation potential and the plate deflection. The first equation is the wave equation, the second is the impermeability condition, the third is the condition of perturbation damping in the gas far from the plate, the fourth is the equation of plate motion, and the fifth is the boundary conditions on the plate edges (F_j are differential operators that determine two conditions at each edge). The dimensionless parameters are

$$\mathrm{M} = \frac{u \cos\theta}{a}, \quad \mathrm{M}_w = \frac{\sqrt{\sigma/\rho_m}}{a}, \quad D = \frac{D_w}{a^2 \rho_m h^3} \quad L = \frac{L_w}{h}, \quad \mu = \frac{\rho}{\rho_m}$$

Here, u and ρ are the gas velocity and density, a is the speed of sound in the gas, σ is the tensile stress in the mid-plane of the plate, ρ_m and h are the material density and the plate thickness, and $D_w = Eh^3/(12(1 - \nu^2))$ is the flexural stiffness of the plate. The parameter M_w is the ratio of the long-wave propagation velocity in the plate to the speed of sound in the gas. The expression for μ is given for the case of unilateral flow; in the case of bilateral flow of the same gas past the plate at the same velocity, the μ value should be doubled. The parameters M, M_w, D, and μ are independent of the plate thickness. We will assume that $\mathrm{M} > 1$ and $L \gg 1$.

We note that in real systems $\mu \ll 1/\sqrt{D}$. For example, for a steel plate ($\rho_m = 8500$ kg/m^3, $E = 2 \cdot 10^{11}$ N/m^2, and $\nu = 0.3$) in an air flow under normal conditions ($\rho = 1$ kg/m^3 and $a = 300$ m/s) we obtain $D \approx 23.9$ and $\mu \approx 1.2 \cdot 10^{-4}$. This makes it possible, in solving problem (1.1), to regard μ as a small parameter, as we shall do below.

2. GLOBAL INSTABILITY OF ONE-DIMENSIONAL SYSTEMS

In [12; 13, §65], sufficient instability conditions were obtained for the homogenous states of one-dimensional, highly extended systems of general form. Two types of instability were found: unilateral, determined by the boundary conditions at one end of the system, and global, independent of the boundary conditions. The boundary conditions traditionally used for a plate: $\partial w/\partial x = w = 0$ at a fixed edge, $\partial^2 w/\partial x^2 = w = 0$ at a hinged edge, and $\partial^2 w/\partial x^2 = \partial^3 w/\partial x^3 = 0$ at a free edge do not satisfy the requirement of unilateral instability of plane perturbations.

The global instability criterion is as follows. Consider the dispersion equation of an unbounded (that is, occupying the entire spatial x axis) system $F(k,\ \omega) = 0$, where k is the wave number and ω is the complex perturbation frequency. For sufficiently large $\mathrm{Im}\,\omega$, its solutions $k_j = k_j(\omega)$, numbered in order of decreasing imaginary part, can be divided into two groups in one of which $\mathrm{Im}\,k_j > 0$ $(j = 1,\ldots,\ s)$ while in the other $\mathrm{Im}\,k_j < 0$ $(j = s+1,\ldots,\ N)$, the number of roots in each group being equal to the number of boundary conditions assigned at one of the ends of the finite system. Each root determines a certain branch of the multivalued analytical function $k = k(\omega)$. With decrease in $\mathrm{Im}\,\omega$ the imaginary parts of the roots decrease in the first group and increase in the second, and at a certain ω the following equality holds:

$$\min_{1\le j\le s} \mathrm{Im}\,k_j = \mathrm{Im}\,k_m = \mathrm{Im}\,k_n = \max_{s+1\le j\le N} \mathrm{Im}\,k_j$$

The set of these ω values defines a curve Ω in the complex ω-plane. For a sufficiently long finite system, part of its eigenvalue spectrum lies near this curve, the more often and the closer to it the longer the system [12; 13, §65]. As $L \to \infty$, far from the system boundaries, the eigenfunctions corresponding to these natural frequencies then take the asymptotic form

$$(C_m e^{ik_m(\omega)x} + C_n e^{ik_n(\omega)x})e^{-i\omega t}$$

Here, C_j are certain constants and i is the imaginary unit. For system instability it is sufficient for a part of the curve Ω to lie in the region $\mathrm{Im}\,\omega > 0$. This is the global instability condition.

3. DISPERSION EQUATION FOR AN UNBOUNDED PLATE IN A GAS FLOW

Let us now investigate the system described in Section 1. For plane perturbations of a plate infinite in all directions in a unilateral gas flow, the dispersion equation can be written in the form [14, 15]:

$$(Dk^4 + \mathrm{M}_w^2 k^2 - \omega^2) - \mu \frac{(\omega - \mathrm{M}k)^2}{\sqrt{k^2 - (\omega - \mathrm{M}k)^2}} = 0 \tag{3.1}$$

Here, k is the wave number and ω is the complex perturbation frequency. The first term describes the contribution of the elastic and inertia plate forces and the second the contribution of the aerodynamic forces acting on the plate. For $\mathrm{Im}\,\omega \gg 1$, in the second term the value of the square root should be so chosen that its real part is positive, which follows from the requirement that the gas perturbations to be damped with distance from the plate. For other $\mathrm{Im}\ \omega$ we so choose the square root branch that the solutions $k(\omega)$ of Eq. (3.1) are analytically continuable from the region $\mathrm{Im}\ \omega \gg 1$ along the rays $\mathrm{Re}\,\omega = \mathrm{const}$, while remaining solutions of (3.1).

In order to study the behavior of the roots $k(\omega)$ of Eq. (3.1), we will consider the equation

$$(Dk^4 + \mathrm{M}_w^2 k^2 - \omega^2) + \mu \frac{(\omega - \mathrm{M}k)^2}{\sqrt{k^2 - (\omega - \mathrm{M}k)^2}} = 0 \tag{3.2}$$

Multiplying the left sides of (3.1) and (3.2) gives a polynomial that for any ω has ten roots k, each of which is a root of either (3.1) or (3.2). We will consider in the complex plane ω the cuts separating different branches of the radical in (3.1) and (3.2), that is, the curves specified by the equation

$$k(\omega)^2 - (\omega - \mathrm{M}k(\omega))^2 = \alpha, \qquad \alpha \in \mathbf{R}, \qquad \alpha < 0$$

After numerically constructing the cuts for certain parameters of the problem and taking any ω lying above them, we see that among all the roots of the polynomial Eq. (3.1) is satisfied by four which at large $\operatorname{Im}\omega$ asymptotically approach the roots of the dispersion equation for an unbounded plate in a vacuum ($\mu = 0$) and that each group (see Section 2) contains two roots. In accordance with general ideas and the global instability criterion we need to track the behavior of only these $k(\omega)$ branches.

If a pair $(\omega,\ k)$ satisfies the dispersion equation (3.1), the latter is also satisfied by the pair $(\omega',\ k')$, where the prime denotes reflection about the imaginary axis. Therefore, the curve Ω is symmetrical about the imaginary ω axis, which enables us to restrict ourselves to seeking points on the curve Ω in the right half-plane of the complex plane ω.

4. GLOBAL INSTABILITY OF HIGH-FREQUENCY PERTURBATIONS OF A FINITE-WIDTH PLATE IN A GAS FLOW

In this section, we will assume μ to be a small parameter. We will also assume that $\omega \gg \mu$ and $k \gg \mu$, which makes it possible to eliminate the low-frequency perturbations from consideration. Let $\omega \in \mathbf{R}$ and $\omega > 0$. For $\mu = 0$ the dispersion equation has four roots k, of which two are real and two are pure imaginary and complex conjugate. We will denote the real positive root by $k_2 = k_2(\omega,\ 0)$ and the negative root by $k_3 = k_3(\omega,\ 0) = -k_2$. These roots correspond to eigenfunctions in the form of harmonic waves that travel in opposite directions at the same phase velocity. We will find the increments of the roots for small $\mu \neq 0$, that is, for a plate in a gas flow, and a frequency $\omega + i\delta$ ($\delta > 0$, $\delta \ll 1$)

$$k_j(\omega + i\delta,\ \mu) = k_j + i\delta\frac{\partial k_j}{\partial\omega} + \mu l(k_j)$$

From the dispersion equation (3.1) we obtain

$$l(k_j) = \frac{(\omega - \mathrm{M}k_j)^2}{2k_j(\mathrm{M}_w^2 + 2Dk_j^2)\sqrt{k_j^2 - (\omega - \mathrm{M}k_j)^2}} \tag{4.1}$$

It is easy to show that $\partial k_2/\partial\omega > 0$ and $\partial k_3/\partial\omega < 0$. Then, if $\operatorname{Im} l(k_2) < \operatorname{Im} l(k_3)$, for any fixed small μ we can find a δ such that $\operatorname{Im} k_2(\omega + i\delta,\ \mu) = \operatorname{Im} k_3(\omega + i\delta,\ \mu)$, which signifies global instability. Thus, the inequality $\operatorname{Im} l(k_2) < \operatorname{Im} l(k_3)$ is a sufficient condition for instability; substituting (4.1) in this inequality and renaming $k_2 = k$ and $k_3 = -k$, we obtain

$$\operatorname{Im}\left(\frac{(\omega + \mathrm{M}k)^2}{\sqrt{k^2 - (\omega + \mathrm{M}k)^2}} + \frac{(\omega - \mathrm{M}k)^2}{\sqrt{k^2 - (\omega - \mathrm{M}k)^2}}\right) < 0 \tag{4.2}$$

Thus, we need to find the parameter values for which inequality (4.2) is satisfied for at least one real positive ω of an order higher than μ.

The physical meaning of sufficient instability condition (4.2) is as follows. Let us first consider an infinitely wide plate and imagine that along it a harmonic wave $w(x,\ t) = e^{i(kx - \omega t)}$ travels at a phase velocity $c = \omega/k = \sqrt{\mathrm{M}_w^2 + Dk^2}$ ($\omega,\ k \in \mathbf{R}$). It can be shown that the pressure perturbation generated by the gas flow past the wave is given by

$$p(x,\ t) = \mu\frac{(\omega - \mathrm{M}k)^2}{\sqrt{k^2 - (\omega - \mathrm{M}k)^2}}e^{i(kx - \omega t)} \tag{4.3}$$

In its turn, the pressure disturbance produces a change in the wave number, which is determined by formula (4.1). As can be seen from (4.1) and (4.3), the spatial growth or damping of the wave, determined by the sign of Im $l(k)$, depends on the nature of the gas flow relative to the wave. If the flow is subsonic,

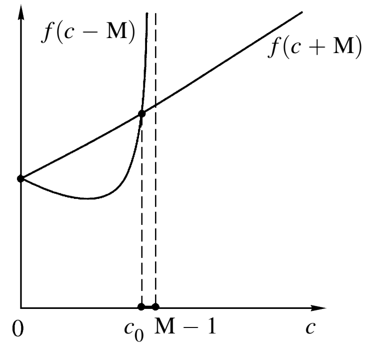

Fig. 2. Graphs of the functions $f(c - \mathrm{M})$ and $f(c + \mathrm{M})$

that is, $|\mathrm{M} - c| < 1$, then the pressure perturbation is in phase with the plate deflection and does not lead to wave growth or damping. If, on the other hand, the flow is supersonic, the pressure is shifted in phase by $\pi/2$ relative to the wave, which leads to the appearance of an imaginary part of the wave number k. In this case, the spatial damping or amplification of the wave depends on the direction of gas flow relative to the wave and on the direction of motion of the wave itself. If these directions coincide, the work done by the gas is positive and the wave is amplified; otherwise the work is negative and the wave is damped. Thus, an upstream-propagating wave always experiences flow resistance and is damped, whereas a downstream-propagating wave is amplified if the gas moves more rapidly than the wave ($\mathrm{M} - c > 1$) and is damped if the wave leaves the flow behind ($\mathrm{M} - c < -1$).

Let us now consider a plate of large but finite length. The mechanism of formation of its global eigenfunction constructed according to [12; 13, §65] is as follows. A wave that travels along the plate in the gas-flow direction is reflected from its rear edge and generates a backward wave. The latter is, in its turn, reflected from the leading edge and generates a forward wave with generally another amplitude. As a result of this process of interconversion of the two waves reflected from the plate edges, a global eigenfunction is formed. If we take the amplitude of the initial wave as unity, the amplitude of the reflected wave will be equal to $A_3 e^{-L\,\mathrm{Im}\,k_2}$ and the amplitude of the doubly reflected wave to $A_3 e^{-L\,\mathrm{Im}\,k_2} A_2 e^{L\,\mathrm{Im}\,k_3}$. Here, A_2 and A_3 are the coefficients of reflection from the leading and rear edges of the plate. Then, for large L, the condition of increase in amplitude on double reflection (i.e., the condition of growth of the global eigenfunction) is given by the inequality $\mathrm{Im}(k_2 - k_3) = \mathrm{Im}(l(k_2) - l(k_3)) < 0$, which is equivalent to (4.2).

We will now investigate inequality (4.2). It is satisfied if both the first and second terms are complex, i.e., the gas flow is supersonic relative to both up- and downstream traveling waves. The signs of the square roots in (4.2) are determined by the choice of $k(\omega)$ branches, which should be so made that at large $\mathrm{Im}\,\omega$ these are roots of the dispersion equation (3.1). On the other hand, the above considerations show that for physical reasons the branches should be chosen as follows: the imaginary part of the first radical in (4.2) must always be negative and that of the second negative for $\mathrm{M} - c < -1$ and positive for $\mathrm{M} - c > 1$. A numerical analysis of the behavior of solutions of (3.1) with variation of Im ω completely confirms this choice. Thus, inequality (4.2) is satisfied only if $\mathrm{M} - c > 1$. In this case, it can be written as follows:

$$f(c + \mathrm{M}) < f(c - \mathrm{M}) \tag{4.4}$$

Here, $f(x) = x^2/\sqrt{x^2 - 1}$, the square root value being chosen positive. Inequality (4.4) is satisfied for $c_0 < c < \mathrm{M} - 1$, where c_0 is the abscissa of the intersection of the graphs (Fig. 2). Thus, the instability condition holds if $c(k) = \sqrt{\mathrm{M}_w^2 + Dk^2} < \mathrm{M} - 1$ for a certain k. Taking into account the fact that k must be of higher order than μ, we find that the high-frequency perturbations are globally unstable if $\mathrm{M} > \mathrm{M} + 1 + \chi(D,\, \mathrm{M}_w,\, \mu)$, where $\chi > 0$ is a small correction whose exact value can be found numerically directly from Eq. (3.1).

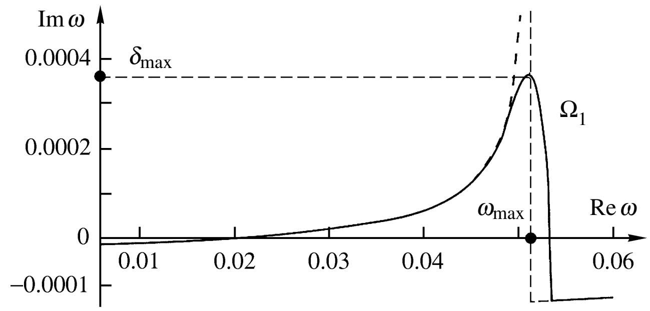

Fig. 3. Part of the curve Ω_1 calculated for parameters (5.4). The continuous curve $k_2(\omega,\ \mu)$ is calculated from (3.1) and the broken curve from (5.3)

5. GROWTH RATE OF HIGH-FREQUENCY PERTURBATIONS

Assuming that the system lies in the region of global instability of high-frequency perturbations obtained in the previous section, we will find the shape of the curve Ω near which the natural frequencies of the system are located (Section 2). For $\omega \gg \mu$ it is determined by the condition

$$\mathrm{Im}\, k_1 > \mathrm{Im}\, k_2 = \mathrm{Im}\, k_3 > \mathrm{Im}\, k_4 \tag{5.1}$$

As will be shown in Section 6, in the low-frequency region the curve Ω may be determined by another expression; therefore, to be specific, we will denote the part given by (5.1) by Ω_1. Assuming the frequency ω to be real and positive, we obtain the following expansion of the wave number in the frequency increment $i\delta$:

$$k_2(\omega + i\delta,\ \mu) = k_2(\omega,\ \mu) + i\delta \frac{\partial k_2(\omega,\ 0)}{\partial \omega}$$

$$k_3(\omega + i\delta,\ \mu) = k_3(\omega,\ \mu) + i\delta \frac{\partial k_3(\omega,\ 0)}{\partial \omega}$$

Observing that $\partial k_2(\omega,\ 0)/\partial\omega = -\partial k_3(\omega,\ 0)/\partial\omega$ and using (5.1), we find

$$\delta(\omega) = -\frac{1}{2}\left(\frac{\partial k_2(\omega,\ 0)}{\partial \omega}\right)^{-1} \mathrm{Im}(k_2(\omega,\ \mu) - k_3(\omega,\ \mu)) \tag{5.2}$$

Expression (5.2) determines the curve Ω_1 consisting of points $\omega + i\delta(\omega)$, where $\omega \in \mathbf{R}$ and $\omega \gg \mu$.

From the results of Section 4 (Fig. 2) it follows that the inequality $\delta(\omega) > 0$ can be satisfied only in the neighborhood of the point $\omega_{\max}$ determined by the equality

$$c(\omega_{\max}) = \mathrm{M} - 1 \ \Rightarrow\ \omega_{\max} = (\mathrm{M} - 1)\sqrt{((\mathrm{M} - 1)^2 - \mathrm{M}_w^2/D}$$

However, in this neighborhood, for small but finite μ we cannot use the expansion

$$k_2(\omega,\ \mu) = k_2(\omega,\ 0) + \mu l(k_2(\omega,\ 0)) \tag{5.3}$$

where $l(k_2(\omega,\ 0))$ can be found from (4.1), since as $c(\omega) \to \mathrm{M} - 1$, $l(k_2(\omega,\ 0)) \to -i\infty$ and approximation (5.3) does not hold. Physically, this is a consequence of a "resonance" between the waves propagating in the plate and in the gas. For fixed parameter values, the curve Ω_1 can be plotted and the maximum growth increment (damping coefficient with the opposite sign) of the high-frequency eigenfunctions $\delta_{\max} = \max\limits_{\omega \in \mathbf{R},\ \omega \gg \mu} \delta(\omega)$ can be numerically calculated by solving the dispersion equation (3.1) for k and substituting

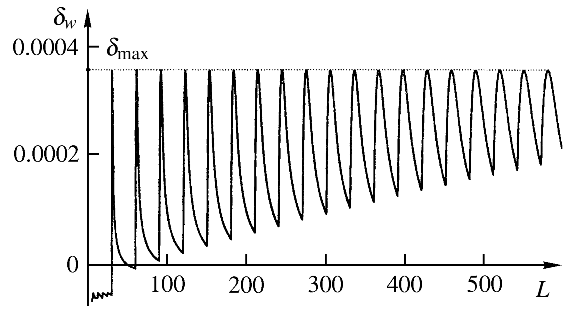

Fig. 4. Dependence $\delta_w(L)$ for parameters (5.4)

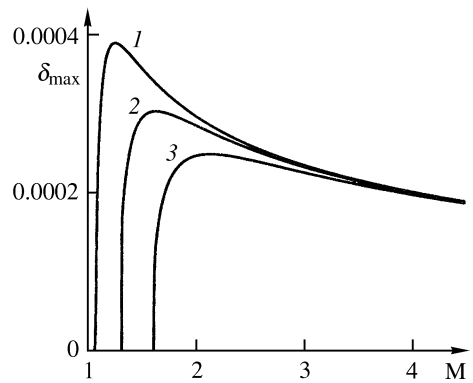

Fig. 5. Dependence $\delta_{max}(\mathrm{M})$ for $D = 23.9$, $\mu = 1.2 \cdot 10^{-4}$, and $\mathrm{M}_w = 0, 0.3, 0.6$ (curves *1–3*)

the values obtained in (5.2). In Fig. 3, as an example, the continuous curve represents part of Ω_1 for the parameters

$$\mathrm{M} = 1.5, \quad \mathrm{M}_w = 0, \quad D = 23.9, \quad \mu = 1.2 \cdot 10^{-4} \tag{5.4}$$

whence we find $\delta_{max} \approx 3.6 \cdot 10^{-4}$. For comparison, the same curve calculated using expression (5.3) is shown as a broken line. All the points of the curve Ω_1 located outside the region of the ω plane shown in the picture have negative imaginary parts. For small Mach numbers ($\mathrm{M} \sim 1.1$) the point of maximum growth increment is displaced to the right of ω_{max} although still close to it; in this respect, the notation ω_{max} is conventional.

We will now consider the discrete spectrum of the natural frequencies of the system. In the absence of a gas the natural frequencies ω_n lie on the real axis ω and $\omega_n \sim (n/L)^2$ as $L \to \infty$. In the presence of a gas, in the first approximation for large L, the natural frequencies satisfying the condition $\omega \gg \mu$ lie on the curve Ω_1 and have the form $\omega_n + i\delta(\omega_n)$, where $\delta(\omega_n)$ is determined by formula (5.2); with increase in the plate width L the frequencies move along the curve Ω_1 in the direction of decreasing real part. As a result, the next dependence of the eigenfunction growth increment on the plate width is formed.

For a sufficiently small width L (we assume that the global stability theory is applicable to the L values considered), $\omega_n \gg \omega_{max}$ for any n and the plate position is stable, since $\delta(\omega_n) < 0$. With increasing L the first natural frequency ω_1 approaches ω_{max} (Fig. 3), enters the flutter zone ($\delta(\omega_1) > 0$), passes through the maximum growth point, and moves away into the stability region again. Then the frequencies ω_2, ω_3, etc. follow the same path. Since the distance between adjacent frequencies lying in the neighborhood of ω_{max} tends to zero as $L \to \infty$, starting from a certain L the plate always lies in the region of high-frequency flutter and with increasing L the growth increment $\delta_w = \max\limits_{n:\,\omega_n \gg \mu} \delta(\omega_n)$ maximal among all high-frequency eigenfunctions displays damped oscillations and asymptotically approaches δ_{max}.

In Fig. 4, for example, the dependence of δ_w on the width of a hinged plate ($\omega_n = (\pi n/L) \times \sqrt{D(\pi n/L)^2 + M_w^2}$) is shown for the parameter values (5.4). With increase in L each next local δ_w minimum corresponds to the transition of maximum-rate growth to the next mode of natural oscillations.

Similarly, we can trace the dependence of the natural frequencies $\omega \gg \mu$ on the Mach number M. Here the natural frequencies of the plate oscillation in a vacuum ω_n remain stationary, while the curve Ω_1 is displaced to the right and deformed. The dependence $\delta_w(\mathrm{M})$ is also oscillatory, but the maximum growth increment $\delta_{\max}$ varies with M (Fig. 5).

The qualitative nature of the dependence $\delta_{\max}(\mathrm{M})$ (Fig. 5) makes it possible to trace the dependence of the maximum growth increment on the angle of incidence θ (Fig. 1). For any M_w there exists an $\mathrm{M} = \mathrm{M}_{\max}(\mathrm{M}_w, D, \mu)$ such that the maximum eigenfunction growth increment $\delta_{\max}$ is maximal for all M. From this it follows that for fixed parameters $\mathrm{M}^* > \mathrm{M}_{\max}$ ($\mathrm{M}^* = u/a$ is the Mach number calculated with respect to the total velocity vector), M_w, D, and μ and for variable incidence angle θ the perturbations independent of the y coordinate grow most rapidly at $\theta = \arccos(\mathrm{M}_{\max}/\mathrm{M}^*)$.

6. GLOBAL INSTABILITY OF LOW-FREQUENCY PERTURBATIONS

We will now consider the case of low-frequency perturbations when in (3.1) the second term is not small as compared with the first.

We will first reproduce the results of a numerical investigation of the behavior of $k(\omega)$ at small ω for the parameter values (5.4). For sufficiently large ω, as follows from the previous sections, the curve Ω is determined by condition (5.1) and consists of points on Ω_1. Let us move along Ω_1 in the direction of decreasing $\mathrm{Re}\,\omega$. Then (5.1) is satisfied only up to a certain ω determined by the condition $\mathrm{Im}\,k_1 > \mathrm{Im}\,k_2 = \mathrm{Im}\,k_3 = \mathrm{Im}\,k_4$ and corresponding to the end of the curve Ω_1. As $\mathrm{Re}\,\omega$ increases further the position of the branches k_3 and k_4 relative to each other changes and the points of the curve Ω are determined by the condition

$$\mathrm{Im}\,k_1 > \mathrm{Im}\,k_2 = \mathrm{Im}\,k_4 > \mathrm{Im}\,k_3 \tag{6.1}$$

We will denote the curve defined by (6.1) by Ω_2. Then Ω is the sum of Ω_1 and Ω_2 and has a break at their junction point. The physical significance of the change in the relative position of $\mathrm{Im}\,k_3$ and $\mathrm{Im}\,k_4$ consists in the following. In the absence of a gas, at small real ω, the amplitude of the wave corresponding to the branch k_3 is spatially constant while the amplitude corresponding to k_4 is slowly damped. Under the action of the gas, as shown in Section 4, the wave corresponding to k_3 is also damped, and since at small ω the wave and the gas interact strongly, the damping is of the same order as the damping of the wave corresponding to k_4. The replacement of $\mathrm{Im}\,k_3 > \mathrm{Im}\,k_4$ by the opposite inequality $\mathrm{Im}\,k_3 < \mathrm{Im}\,k_4$ indicates that at small frequencies, under the action of the gas, the wave k_3 is damped more strongly than the wave k_4 and the global eigenfunctions correspond to the waves k_2 and k_4, not k_2 and k_3.

Let us consider, for the parameter values (5.4), the structure of the level curves of the function $\mathrm{Re}\,\omega(k)$ calculated numerically and shown in Fig. 6. In the fourth quadrant of the k plane there is a branch point $k^* = k_2 = k_4 = 0.0118 - 0.0068i$ which corresponds to $\omega^* = 0.0013 + 7.72 \cdot 10^{-4} i$ and is the end of Ω_2. The calculated curve Ω_2 and the part of Ω_1 that lies in the region of small ω are shown in Fig. 7. For the parameter values (5.4), since $\mathrm{Im}\,\omega^* > 0$, the low-frequency perturbations are globally unstable and, since $\mathrm{Im}\,\omega^*$ is more than twice as great as $\delta_{\max}$ (Fig. 3), the low-frequency flutter is more intense than the high-frequency one.

Now let the values of the problem parameters be arbitrary. The numerical investigation of the behavior of $k_j(\omega)$ shows that the point of branching of k_2 and k_4 is the only possible point of branching of the roots of the different groups in the region $\mathrm{Im}\,\omega > 0$. If this branch point exists, then the maximum of the imaginary parts of ω lying on the curve Ω_2 is reached at its end, at the branch point. We will investigate its position in the complex plane ω. Assuming that $|k| \gg |\omega|$, we simplify the dispersion equation (3.1) by neglecting ω in the second term and choosing a definite branch of the square root:

$$Dk^4 + \mathrm{M}_w^2 k^2 - \omega^2 + i\mu \frac{\mathrm{M}^2}{\sqrt{\mathrm{M}^2 - 1}} k = 0 \tag{6.2}$$

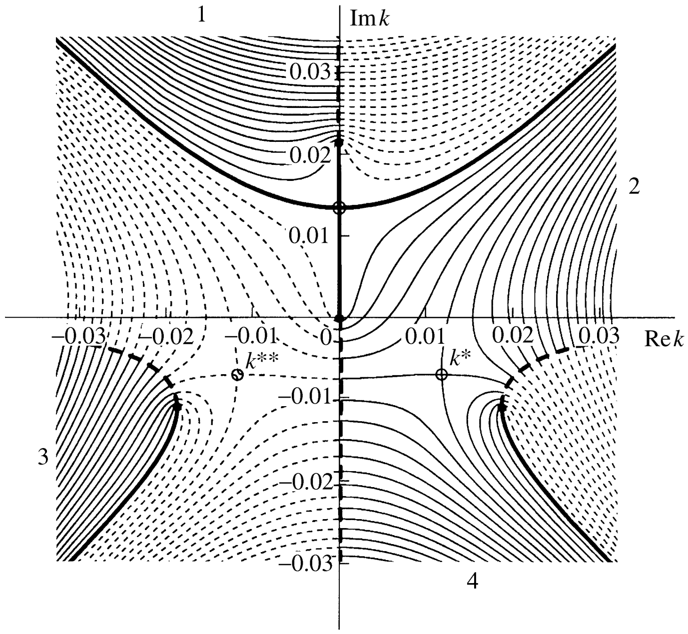

Fig. 6. Level lines $\mathrm{Re}\,\omega(k)$ in the region of small k. The originals of the rays $\mathrm{Re}\,\omega = \mathrm{const} > 0$ are shown for $\mathrm{Im}\,\omega > 0$ by continuous and for $\mathrm{Im}\,\omega < 0$ by broken lines. The originals of $\mathrm{Re}\,\omega = 0$ are shown by heavy continuous and those of $\mathrm{Im}\,\omega = 0$, $\mathrm{Re}\,\omega > 0$ by heavy broken lines. The originals of $\omega = 0$ are shown by dots and the branch points by circles. The numbers *1–4* denote the branches $k_j(\omega)$ which map the region $\mathrm{Re}\,\omega > 0$, $\mathrm{Im}\,\omega > 0$ into the region containing the corresponding number

The branch points (6.2) are determined by the system

$$\begin{gathered} 4Dk^3 + 2\mathrm{M}_w^2 k + i\mu\frac{\mathrm{M}^2}{\sqrt{\mathrm{M}^2-1}} = 0 \\ 2\mathrm{M}_w^2 k^2 + 3i\mu\frac{\mathrm{M}^2}{\sqrt{\mathrm{M}^2-1}}k = 4\omega^2 \end{gathered} \tag{6.3}$$

The first equation in (6.3) was obtained by differentiating (6.2) with respect to k and the second by subtracting the first equation (6.3) multiplied by $k/4$ from (6.2).

We will first consider the case $\mathrm{M}_w = 0$. Then one of the solutions of (6.3) takes the form:

$$k^* = \left(\mu\frac{\mathrm{M}^2}{\sqrt{\mathrm{M}^2-1}}\right)^{1/3}(4D)^{-1/3}e^{-i\pi/6}, \quad \omega^* = \frac{\sqrt{3}}{2}\left(\mu\frac{\mathrm{M}^2}{\sqrt{\mathrm{M}^2-1}}\right)^{2/3}(4D)^{-1/6}e^{i\pi/6} \tag{6.4}$$

For small μ, values (6.4) satisfy the condition $|k| \gg |\omega|$ and therefore approximately coincide with one of the branch points of the function $k(\omega)$ obtained by multiplying (3.1) by (3.2). For the parameter values (5.4), expression (6.4) yields the values $k^* = 0.0118 - 0.0068i$ and $\omega^* = 0.0013 + 7.85 \cdot 10^{-4}i$ which are close to the required branch point of k_2 and k_4 (Fig. 6). From this it follows that, firstly, at $\mathrm{M}_w = 0$, (6.4) also gives the required branch point for other values of the parameters M, D, and μ and, secondly, if the condition $|k| \gg |\omega|$ is satisfied, then at $\mathrm{M}_w \neq 0$, among all the solutions of (6.3), the approximation to the required branch point is given by the solution which can be obtained from (6.4) by a continuous transformation with M_w varying from zero to a fixed value. From (6.4) it can be seen that for $\mathrm{M}_w = 0$ the branch point (k^*, ω^*) always lies in the region $\mathrm{Im}\,\omega > 0$ and thus the low-frequency perturbations are globally unstable. System (6.3) also admits another solution (k^{**}, ω^{**}) differing from (6.4) in that k is reflected about the imaginary and ω about the real axis. This branch point is of "mixed" type in the sense that at this point the root k_3 merges with the branch formed during the branching of roots k_2 and k_4 at point (6.4) (Fig. 6). The third

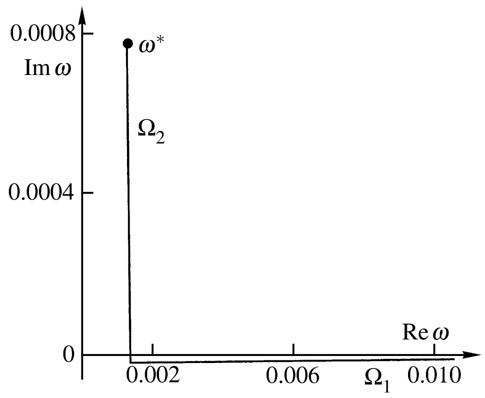

Fig. 7. Curve Ω_2 and the part of the curve Ω_1 that lies in the region of small ω, calculated for parameters (5.4)

solution of (6.3) gives a branch point of the roots k_1 and k_2 which belong to the same group; this point is unrelated to global instability and will not be considered in what follows.

We will study the change in the position of the branch points with increasing M_w. For

$$M_w < \left(\frac{\sqrt{54}}{4}\right)^{1/3} \left(\mu \frac{M^2}{\sqrt{M^2-1}}\right)^{1/3} D^{1/6}$$

by applying the argument principle to the first of equations (6.3), it can be shown that at the branch point $k^*(M_w)$ obtained by continuing (6.4) the inequalities $\mathrm{Re}\,k^* > 0$ and $\mathrm{Im}\,k^* < 0$ are satisfied and $\mathrm{Im}\,k^*(M_w)$ monotonously decreases with increase in M_w. For these conditions, from the second equation it follows that ω^2 cannot be real and, hence, $\mathrm{Re}\,\omega^* > 0$ and $\mathrm{Im}\,\omega^* > 0$, that is, the low-frequency perturbations remain globally unstable. As before, the branch point $(k^{**},\ \omega^{**})$ is the mirror reflection of the point $(k^*,\ \omega^*)$ about the imaginary k and real ω axes.

For

$$M_w = \left(\frac{\sqrt{54}}{4}\right)^{1/3} \left(\mu \frac{M^2}{\sqrt{M^2-1}}\right)^{1/3} D^{1/6} \tag{6.5}$$

the branch points $(k^*,\ \omega^*)$ and $(k^{**},\ \omega^{**})$ merge into one triple branch point

$$k^* = k^{**} = -\frac{i}{2}\left(\mu \frac{M^2}{\sqrt{M^2-1}}\right)^{1/3} D^{-1/3}, \quad \omega^* = \omega^{**} = -\frac{\sqrt{3}}{4}\left(\mu \frac{M^2}{\sqrt{M^2-1}}\right)^{2/3} D^{-1/6}$$

at which the roots k_2, k_3, and k_4 coincide. Value (6.5) lies on the boundary of the global instability region, since the merging takes place on the real axis ω.

With further increase in M_w the merged branch points separate, moving along the imaginary axis in the k plane and along the real axis in the ω plane, that is, perpendicular to their previous trajectories. The roles of these points change. At the first branch point, which moves in the direction of decreasing $\mathrm{Im}\,k$, the roots k_3 and k_4 merge, which can be established by numerically constructing for any problem parameters level curves similar to those in Fig. 6. Since the roots belong to the same group, this point is unrelated to global instability. The second point moves in the direction of increasing $\mathrm{Im}\,k$ and decreasing $\mathrm{Re}\,\omega$ and is a branch point of the roots k_2 and k_3. For sufficiently large M_w the condition $|k| \gg |\omega|$ is not satisfied and (6.2) poorly approximates the dispersion equation (3.1). However, a numerical investigation shows that the branch point of the roots k_2 and k_3 calculated from (3.1) is displaced from the real axis ω into the lower half-plane and does not lead to instability. This point is the end of the curve Ω_1 which thus can depart into the upper ω half-plane only in the region of high-frequency perturbations. Since in this case the curve Ω_2 does not contain any points, if M_w exceeds the value (6.5), then the low-frequency perturbations are not globally unstable.

7. DISCUSSION OF THE RESULTS

We will first make an observation concerning the applicability of approximate hydrodynamic theories to plate flutter problems. In fact, in our study, for finding the pressure acting on an oscillating plate the following method was used: the oscillation was represented as a superposition of traveling waves and the pressure was assumed to be a superposition of pressures (4.3) acting on these waves. By using the exact deflection dependence of the pressure [1, 3, 6], it can be shown that for Im $\omega > 0$, as $L \to \infty$, this method is asymptotically exact. Let us assume that, instead of the exact expression (4.3), for the pressures acting on the traveling waves the following dependence of the pressure on the plate deflection is used:

$$p(x, t) = C_1 \frac{\partial \omega}{\partial x} + C_2 \frac{\partial \omega}{\partial t} \tag{7.1}$$

where C_j are functions of the problem parameters and $C_2 \geq 0$. In particular, a dependence of this form is given by the piston theory and by some other approximations of the exact dependence for large Mach numbers. Then the analog of inequality (4.2) takes the form $\mathrm{Im}(i(C_1k + C_2\omega) + i(-C_1k + C_2\omega)) < 0$ and the entire curve Ω_1 lies in the region Im $\omega \leq 0$. Thus, the high-frequency flutter cannot be obtained using (7.1); this seems to be the main disadvantage of such dependences. We note that the theory developed in [10] correctly describes the high-frequency flutter and gives a criterion identical to that obtained in this study.

The low-frequency flutter is correctly described by dependences of the form (7.1), since it was investigated using for the dispersion equation approximation (6.2), which can be treated as the exact dispersion equation obtained using a quasi-static approximation of form (7.1), where $C_1 = \mu \mathrm{M}^2/\sqrt{\mathrm{M}^2 - 1}$ and $C_2 = 0$.

We will now explain the difference between high- and low-frequency flutter. In [5, 16], two types of plate flutter are described. The first, with a single degree of freedom, arises as a result of one of the natural frequencies entering the region Im $\omega > 0$ under the action of negative aerodynamic damping of the corresponding plate oscillation mode. In this case, there is no interaction between the oscillation modes. The other type is coupled flutter, which is a result of the interaction between two oscillation modes. Let us consider the destabilization mechanism for this flutter type [2, 3, 9]. For a sufficiently small Mach number or plate width, all the natural frequencies have the same negative imaginary part. With increase in M or L the first and second natural frequencies move toward each other and their trajectories are parallel to the real axis ω. At a certain moment they meet and then drift apart in directions perpendicular to the initial ones: one frequency moves in the direction of decreasing and the other of increasing imaginary part. As the latter intersects the real axis, destabilization occurs.

We will prove that the high- and low-frequency flutters obtained in this study are flutter with a single degree of freedom and coupled flutter, respectively. We will first consider the high-frequency flutter. It is easy to see that the structure of the increasing high-frequency eigenfunction described in Section 4 is identical to the structure of the eigenfunctions for a plate in a vacuum. The difference lies in the fact that after two reflections in a vacuum the wave amplitude coincides with the initial amplitude, whereas in a gas it increases, which results in the growth of the eigenfunction with time. Thus, the high-frequency flutter oscillations arise due to negative aerodynamic damping of one or more oscillation modes and occur without any intermodal interaction.

From the results obtained in [2, 3, 9] within the framework of piston theory it follows that a plate in the form of a sufficiently wide strip lies in the region of coupled flutter. Since high-frequency flutter cannot be obtained in the piston theory approximation, the coupled flutter is a low-frequency flutter. Moreover, using the results obtained in [2], it can be shown that the trajectories along which the frequencies of the time-increasing eigenfunctions move with increase in L are qualitatively similar to the curve Ω_2 (Fig. 7). Hence, the concepts of low-frequency and coupled flutter coincide.

This correspondence between flutter types explains why flutter with a single degree of freedom was never described by Soviet and Russian authors. The vast majority of studies used relations of (7.1) type, which, as shown above, do not describe flutter with a single degree of freedom. On the other hand, in the studies that used more correct expressions for the dependence of the pressure on the deflection, in the numerical solution, the number of series terms retained was insufficient to obtain high-frequency flutter. We note that in handbooks on aircraft design [17–19] the panel flutter criterion is obtained using quasi-static and piston theory and therefore flutter with a single degree of freedom is not excluded.

Summary. The global instability of a wide elastic strip in a gas flow is studied. It is shown that there are two types of instability corresponding to the instability of high- and low-frequency perturbations, respectively. Both are oscillatory and thus of flutter type.

The high-frequency flutter criterion and the frequency at which the eigenfunctions increase most intensely have the approximate form:

$$\mathrm{M} > \mathrm{M}_w + 1, \quad \omega = (\mathrm{M}-1)\sqrt{((\mathrm{M}-1)^2 - \mathrm{M}_w^2)/D}$$

The high-frequency flutter is a consequence of negative aerodynamic damping of the natural oscillations of the plate. The mechanism of formation of increasing eigenfunctions consists of the cyclic reflections and interconversions of two waves traveling in opposite directions, during which their amplitudes increase.

The structure of the system's high-frequency oscillation spectrum is described. With increase in the plate width L, the L dependence of the increment of the most rapidly increasing high-frequency eigenfunction is oscillatory in nature. As $L \to \infty$, the oscillations are damped, approaching the maximum possible increment, and the number of the most rapidly growing eigenfunction increases monotonically. The dependence of the growth increment of the most rapidly increasing high-frequency eigenfunction on the Mach number M is found.

The low-frequency flutter criterion and frequency have approximately the form:

$$\mathrm{M}_w < \left(\frac{\sqrt{54}}{4}\right)^{1/3} \left(\mu \frac{\mathrm{M}^2}{\sqrt{\mathrm{M}^2-1}}\right)^{1/3} D^{1/6}, \quad \omega = A\left(\mu \frac{\mathrm{M}^2}{\sqrt{\mathrm{M}^2-1}}\right)^{2/3} D^{-1/6}$$

where A depends on the parameters of the problem and varies from 0.433 on the boundary of the flutter region up to 0.595 at $\mathrm{M}_w = 0$. Low-frequency flutter arises as a result of the interaction of oscillation modes. It can be adequately described using the quasi-static dependence of the gas pressure on the plate deflection, whereas the high-frequency flutter cannot be obtained either by means of quasi-static and piston theory or by using any approximation of the form (7.1).

The author is grateful to A. G. Kulikovskii for numerous discussions of the problems considered.

The work was supported financially by the Russian Foundation for Basic Research (project No. 05-01-00219) and by a Presidential Grant for the Support of Leading Scientific Schools (project No. NSh-1697.2003.1).

REFERENCES

1. H. C. Nelson and H. J. Cunningham, *Theoretical Investigation of Flutter of Two-Dimensional Flat Panels with One Surface Exposed to Supersonic Potential Flow*, NACA, Rep. No. 1280 (1956).[1]
2. A. A. Movchan, "On the oscillations of a plate moving in a gas," *Prikl. Mat. Mekh.*, **20**, No. 2, 211-222 (1956).
3. V. V. Bolotin, *Nonconservative Problems of Elastic Stability Theory* [in Russian], Fizmatgiz, Moscow (1961).
4. E. I. Grigolyuk, R. E. Lamper, and L. G. Shandarov, "Flutter of panels and shells", *Advances in Science and Technology. Mechanics* [in Russian], VINITI, Moscow, 34–90 (1965).
5. E. H. Dowell, "Panel flutter: A review of the aeroelastic stability of plates and shells," *AIAA J.*, **8**, No. 3, 385–399 (1970).
6. T. Y. Yang, "Flutter of flat finite element panels in supersonic potential flow," *AIAA J.*, **13**, No. 11, 1502–1507 (1975).

[1] The paper is available at http://naca.larc.nasa.gov/reports/1956/naca-report-1280/naca-report-1280.pdf

7. Yu. N. Novichkov, "Flutter of plates and shells", *Advances in Science and Technology. Mechanics of Deformable Solids* [in Russian], **11**, VINITI, Moscow, 67–122 (1978).
8. A. A. Ilyushin and I. A. Kiiko, "Oscillations of a rectangular plate in a supersonic gas flow," *Vestn. MGU, Ser. 1, Mathematics, Mechanics*, No. 4, 40–44 (1994).
9. S. D. Algazin and I. A. Kiiko, "Investigation of the operator eigenvalues in panel flutter problems," *Izv. Ros. Akad. Nauk, Mekh. Tverd. Tela*, No. 1, 170–176 (1999).
10. D. M. Minasyan and M. M. Minasyan, "A novel approximation in the problem of plate flutter in a supersonic gas flow," *Dokl. Nats. Akad. Nauk Armenii*, **101**, No. 1, 49–54 (2001).
11. D. M. Minasyan "Calculation of the frequencies of unimodal flutter oscillation of a finite plate," *Izv. Nats. Akad. Nauk Armenii, Ser. Mekhanika*, **54**, No. 4, 26–33 (2001).
12. A. G. Kulikovskii, "On the stability of homogeneous states," *Prikl. Mat. Mekh.*, **30**, No. 1, 148–153 (1966).
13. E. M. Lifshits and L. P. Pitaevskii, *Theoretical Physics. V. 10. Physical Kinetics* [in Russian], Nauka, Moscow (1979).
14. A. Kornecki, "Aeroelastic and hydroelastic instabilities of infinitely long plates. II," *Solid Mech. Archives*, **4**, No. 4, 241–346 (1979).
15. V. V. Vedeneev, "Instability of an unbounded elastic plate in a gas flow," *Fluid Dynamics*, **39**, No. 4, 526–533 (2004).
16. *Panel Flutter. NASA Space Vehicle Design Criteria (Structures)*, NASA SP-8004 (1972).[2]
17. S. N. Kan and I. A. Sverdlov, *Aircraft Strength Calculations* [in Russian], Mashinostroenie, Moscow (1966).
18. *Vibration in Technology. Handbook in 6 Volumes. V. 3. Vibration of Mechanisms, Structures and their Elements* [in Russian], Mashinostroenie, Moscow (1980).
19. A. I. Smirnov, *Aeroelastic Stability of Aircraft* [in Russian], Mashinostroenie, Moscow (1980).

[2]The paper is available at http://trs.nis.nasa.gov/archive/00000117/01/sp8004.pdf