ISSN 0015-4628, Fluid Dynamics, 2007, Vol. 42, No. 5, pp. 858–868. © Pleiades Publishing, Ltd., 2007.
Original Russian Text © S.V. Guvernyuk, G.Ya. Dynnikova, 2007, published in Izvestiya Rossiiskoi Akademii Nauk, Mekhanika Zhidkosti i Gaza, 2007, Vol. 42, No. 5, pp. 197–208.

Nonlinear High-Frequency Flutter of a Plate

V. V. Vedeneev

Received February 1, 2007

Abstract—In recent studies of the problem of linear stability of a plate in a supersonic gas flow a new ("high-frequency") type of flutter, which cannot be obtained by means of the piston theory usually employed in these problems, was found to exist together with the classical ("low-frequency") type. In the present study a new method of calculating the pressure acting on a high-frequency vibrating plate is proposed and, using this method, high-frequency flutter is investigated in the nonlinear formulation and the flutter vibration amplitudes are determined.

DOI: 10.1134/S0015462807050183

Keywords: nonlinear flutter, nonlinear vibrations, high-frequency flutter, plate flutter, panel flutter.

1. FORMULATION OF THE PROBLEM

We will consider an elastic, isotropically stretched plate exposed on the side to a plane-parallel supersonic inviscid perfect gas flow, while on the other side a constant pressure, which balances the plate, is maintained. The plate is embedded in an absolutely rigid plane separating the gas flow from the constant pressure zone. The problem of the stability of this system was studied in [1–3]. In addition to the classical well-investigated ("low-frequency") type of flutter observed when investigating the problem by means of the piston theory, a new ("high-frequency") type of flutter, which cannot be obtained by means of the piston theory, was found. The present study is devoted to the investigation of high-frequency flutter in the nonlinear formulation and the determination of the flutter vibration amplitudes.

We will assume that the nonlinearity of the problem is due to the geometric nonlinearity of the behavior of the plate, namely, the presence of the membrane stresses developing during bending (Kármán large deflection model). The gas pressure perturbation acting on the plate will be assumed to be linearly dependent on the deflection since the aerodynamic nonlinearity significantly affects the plate vibrations only at very high Mach numbers (of the order of 20) [4, §4.18]. Initially, we will consider the problem in the two-dimensional formulation, the question of the extension of the results to rectangular plates being considered latter. The dimensionless equation of motion of the plate has the form [5, §24]:

$$D\frac{\partial^4 w}{\partial x^4} - \left(M_w^2 + K\frac{1}{2L}\int\limits_{-L/2}^{L/2}\left(\frac{\partial w(\xi)}{\partial \xi}\right)^2 d\xi\right)\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial t^2} - P\{w\} = 0,$$

$$D = \frac{E}{12(1-\nu^2)a^2\rho_m}, \quad M_w = \frac{\sigma\rho_m}{a}, \quad K = \frac{E}{(1-\nu^2)a^2\rho_m} = 12D, \qquad (1.1)$$

$$L = \frac{L_w}{h}, \qquad M = \frac{u}{a}, \qquad \mu = \frac{\rho}{\rho_m}.$$

Here, $w(x,\ t)$ is the deflection of the plate divided by its thickness, E, ν, and ρ_m are Young's modulus, Poisson's ratio, and the density of the plate material, L_w and h are the plate width and thickness, and u, ρ, and a are the velocity, the density, and the sonic speed of the gas. The plate tension (the coefficient of $\partial^2 w/\partial x^2$) consists of from two parts: a constant tension applied to the plate (the first term, σ is the tensile stress) and an

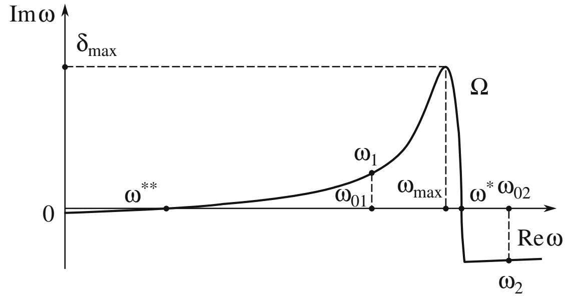

Fig. 1. Part of the curve Ω corresponding to high-frequency flutter. The characteristic scale along the vertical axis is 100 times less than the horizontal scale.

additional nonlinear tension appearing in the deflection of the plate (the second term). The parameters D and L are the dimensionless stiffness and width of the plate, M_w and K characterize its tension and nonlinearity, and M and μ are the Mach number and the dimensionless density of the gas. The expression for K is written on the assumption that the plate edges are not displaced during bending, otherwise K is less than this value.

The operator $P\{w(x,\ t)\}$ is the gas pressure perturbation; in the case of harmonic oscillations it has the form [4, §4.7]:

$$P\{W(x)\cos\omega t\} = -\mathrm{Re}\frac{\mu}{\sqrt{M^2-1}}e^{-i\omega t}\left(-i\omega + M\frac{\partial}{\partial x}\right)\int\limits_{-L/2}^{x}\left(-i\omega W(\xi) + M\frac{\partial W(\xi)}{\partial \xi}\right)\exp\left(\frac{iM\omega(x-\xi)}{M^2-1}\right)J_0\left(\frac{-\omega(x-\xi)}{M^2-1}\right)d\xi. \quad (1.2)$$

For calculating $P\{w\}$ in the case of nonharmonic oscillations it is necessary to represent the deflection in the form of a Fourier series or integral and to use the linearity of the operator P.

The plate occupies the domain $-L/2 \le x \le L/2$ and restraint or hinged conditions must be specified at the edges. We will give the characteristic values of the parameters for a steel plate exposed to an air flow under normal conditions ($E = 2\times10^{11}$ N/m^2, $\nu = 0.3$, $a = 330$ m/s, $\rho = 1$ kg/m^3, and $\rho_m = 8500$ kg/m^3). In order to determine the maximum possible M_w we take $\sigma = \sigma_B = 2\times10^9$ N/m^2, i.e. the ultimate strength of steel and then obtain

$$D = 19.8, \qquad M_w = 1.5, \qquad K = 237, \qquad \mu = 1.2\times10^{-4}. \quad (1.3)$$

In what follows, we will assume that the condition $M > M_w + 1$, which is necessary for the development of high-frequency flutter, is satisfied [1] and the quantity μ will be assumed to be a small parameter.

2. RESULTS OF THE LINEAR STABILITY INVESTIGATION

We will formulate the method of calculating the eigenfrequencies of the linearized problem (1.1) [1, 2]. We will consider the complex plane of the eigenfrequencies ω (the oscillations are assumed to be time-dependent as $e^{-i\omega t}$). In it we will plot the eigenfrequencies ω_{0n} of the plate in a vacuum; all the eigenfrequencies lie on the real number axis. In this plane we will consider the curve Ω, the asymptotic location of the plate spectrum in the gas flow at large L [1], shown in Fig. 1. At points ω^*, ω^{**} it intersects the real axis and lies in the upper half-plane on the interval $\omega^{**} < \mathrm{Re}\,\omega < \omega^*$ and in the lower half-plane outside it. When

$$\mathrm{Re}\,w = \omega_{\max} = (M-1)\sqrt{((M-1)^2 - M_w^2)/D}, \quad (2.1)$$

the imaginary part of the points lying on Ω reaches a maximum:

$$\operatorname{Im}\omega = \delta_{\max} = \mu^{2/3}\frac{\sqrt{3}}{8}\left(\frac{(M-1)^2 - M_w^2}{D}\right)^{1/6} \times \frac{\left(2(M-1)^2 - M_w^2\right)^{1/3}}{(M-1)^{4/3}} - \mu\frac{(2M-1)^2}{4(M-1)\sqrt{(2M-1)^2-1}}.$$

The distance $|\omega^* - \omega_{\max}| \sim \mu^{2/3}$; therefore, $\omega^* \approx \omega_{\max}$. The value of

$$\omega^{**} = \sqrt{(M^2+1-\sqrt{4M^2+1})(M^2+1-\sqrt{4M^2+1}-M_w^2)/D},$$

is real when $M > \sqrt{\sqrt{(4M_w^2+1)} + M_w^2 + 1}$; otherwise, on the entire interval $0 < \operatorname{Re}\omega < \omega^*$ the curve Ω lies in the upper half-plane.

By carrying out these constructions, we can approximately determine the eigenfrequencies ω_n of the plate-gas system as follows: they lie on the curve Ω and their real parts are equal to ω_{0n}. As an example, in Fig. 1 we have reproduced the location of the first two eigenfrequencies. We note that the location of Ω depends only on the parameters M, M_w, D, and μ and is independent of the plate width L and the boundary conditions specified on the plate edges.

Let the Mach number be fairly small (not greater than $M_w + 1$) so that in a vacuum all the plate frequencies $\omega_{0n} > \omega^*$. In this case the system is linearly stable since $\operatorname{Im}\omega_n < 0$. As the Mach number M increases, the curve Ω is displaced to the right; the equality $\omega_{01} = \omega^*$ becomes valid when $M = M^*$. This Mach number lies on the boundary of the stability domain. With further increase in M the lowest oscillation mode starts to grow ($\operatorname{Im}\omega_1 > 0$), i.e., high-frequency flutter develops, continuing while $\omega^{**} < \omega_{01} < \omega^*$. The equality $\omega_{01} = \omega^{**}$ becomes valid when $M = M^{**}$ and the first oscillation mode again starts to be damped when $M > M^{**}$. Analogously, we can trace the behavior of the second and subsequent modes for which high-frequency flutter develops if their frequencies lie on the interval $\omega^{**} < \omega_{0n} < \omega^*$.

With further increase in M this method of calculating the eigenfrequencies ceases to be valid for modes that have passed through the high-frequency instability domain ($\omega_{0n} \ll \omega^{**}$). The action of the flow on these modes becomes significant and $\operatorname{Re}\omega_n \neq \omega_{0n}$; then the eigenfrequencies can interact and depart to the upper half-plane. In this case low-frequency flutter develops, i.e., instability of the type which can be observed in investigating the problem by means of the piston theory [1]. In the present study low-frequency flutter is not considered and it is always assumed that the Mach number is not too high, so that only high-frequency flutter can develop.

3. DERIVATION OF AN EQUATION FOR THE AMPLITUDE

Let there be some solution of Eq. (1.1). We will expand it in terms of the natural modes of plate oscillation in a vacuum:

$$w(x,\, t) = \sum_{j=1}^{\infty} W_j(x) A_j(t), \tag{3.1}$$

with as yet unknown amplitudes $A_j(t)$. This expansion exists and is unique by virtue of the completeness of the plate eigenfunctions.

We will substitute this expansion in (1.1) and use the Bubnov–Galerkin method multiplying (1.1) by $W_n(x)$ and integrating with respect to x from $-L/2$ to $L/2$. By virtue of the orthonormality of the eigenfrequencies the following equalities hold:

$$\int_{-L/2}^{L/2}\left(D\frac{\partial^4 W_j}{\partial x^4} - M_w^2\frac{\partial^2 W_j}{\partial x^2}\right)W_n dx = \omega_{0n}^2\delta_j^n, \qquad \int_{-L/2}^{L/2} W_j W_n\, dx = \delta_j^n.$$

Here, δ_j^n is the Kronecker delta. Setting

$$a_{jn} = \frac{1}{\sqrt{2L}} \int\limits_{-L/2}^{L/2} \frac{\partial W_j}{\partial x} \frac{\partial W_n}{\partial x} dx = -\frac{1}{\sqrt{2L}} \int\limits_{-L/2}^{L/2} \frac{\partial^2 W_j}{\partial x^2} W_n \, dx,$$

$$P_{jn}(t) = \int\limits_{-L/2}^{L/2} P\{W_j A_j\} W_n \, dx,$$

we obtain the following equation for the nth amplitude

$$\frac{\partial^2 A_n}{\partial t^2} + \omega_{0n}^2 A_n + K \sum_{m,k,j=1}^{\infty} a_{mk} a_{jn} A_m A_k A_j - \sum_{j=1}^{\infty} P_{jn} = 0. \tag{3.2}$$

Equations (3.2) are an infinite system of equations for the amplitudes A_n, $n = 1, 2, \ldots.$ In this case $a_{jn} = a_{nj}$, $a_{jj} > 0$, and in the particular case of a hinged support on both edges $a_{jn} = 0$ when $j \neq n$.

As will be shown below, the nonlinear oscillations are similar to harmonic ones; therefore, in order to calculate the pressure we can use formula (1.2). However, in view of the complexity of its direct use we will employ the following approximate method for calculating the pressure.

Initially, we will consider the linearized equations (3.2)

$$\frac{\partial^2 A_n}{\partial t^2} + \omega_{0n}^2 A_n - \sum_{j=1}^{\infty} P_{jn} = 0. \tag{3.3}$$

From the results of linear stability investigations it is known that in the case of high-frequency flutter or stability the natural oscillation modes of a plate in a flow are close to the natural modes of the plate in a vacuum. In other words, by considering the solution of the system of equations (3.3) in the form:

$$A_j(t) = C_j e^{-i\omega t}, \qquad j = 1, 2, \ldots \tag{3.4}$$

we can obtain a denumerable set of values of ω_n such that $|\omega_n - \omega_{0n}| \sim \mu \ll 1$, $|C_j|/|C_n| \sim \mu \ll 1$, $j \neq n$. From the last estimate it follows that $|P_{jn}|/|P_{nn}| \sim \mu \ll 1$, $j \neq n$ and we can replace the series in (3.3) by the term P_{nn}.

We will study the structure of the quantity P_{nn}. From (1.2) it can be seen that for harmonic oscillations ($A_n(t) = e^{-i\omega t}$) we can write $P_{nn}(t) = p_{n1} e^{-i\omega t} + 2p_{n2} e^{-i\omega t}(-i\omega)$, where p_{n1} and p_{n2} are functions of ω. Expressing P_{nn} in terms of A_n, we obtain a more convenient formula which can be used in the nonlinear case

$$P_{nn}(t) = p_{n1}(\omega) A_n(t) + 2p_{n2}(\omega) \frac{\partial A_n(t)}{\partial t}. \tag{3.5}$$

We then substitute expression (3.4) in (3.3), retaining only the basic, nth harmonic, and use (3.5)

$$-\omega_n^2 + 2i\omega_n p_{n2}(\omega_n) + \omega_{0n}^2 - p_{n1}(\omega_n) = 0,$$

$$\omega_n = \sqrt{\omega_{0n}^2 - p_{n1}(\omega_n) - p_{n2}^2(\omega_n)} + ip_{n2}(\omega_n) = \sqrt{\omega_{0n}^2 - p_{n1}(\omega_{0n}) - p_{n2}^2(\omega_{0n})} + ip_{n2}(\omega_{0n}).$$

The last equality is valid correct to small values of the order of μ. Since $p_{nj} \sim \mu \ll 1$, the quantity p_{n2} is the imaginary part of the eigenfrequency, i.e., the oscillation amplification index, and p_{n1} corresponds only to the small variation of the real part of the eigenfrequency under the action of the gas.

Denoting $\delta \equiv \operatorname{Im} \omega$ and $\omega \equiv \operatorname{Re} \omega$, we represent the curve Ω (Fig. 1) as the graph of the dependence $\delta(M, M_w, D, \mu, \omega)$ for given M, M_w, D, and μ. Above we have shown that $p_{n2}(\omega_{0n}) = \delta(M, M_w, D, \mu, \omega_{0n})$.

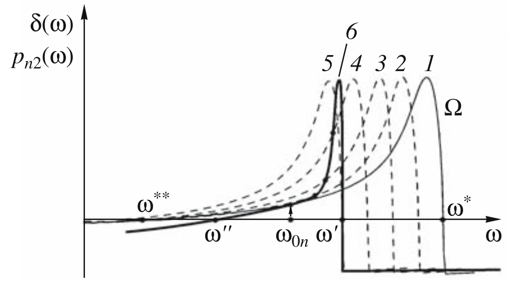

Fig. 2. Construction of the dependence $p_{n2}(\omega)$: curve (*1*) corresponds to Ω, curves (*2–5*) to the same curve displaced to the left with increase in ω, and curve (*6*) to the resultant dependence $p_{n2}(\omega)$.

Now let the *n*th oscillation mode occur at a frequency ω which is different from ω_{0n}. From the standpoint of the gas it is of no importance why the frequency changes—due to nonlinearity or, for example, due to a proportional change in the plate stiffness or tension since the shape of the oscillation $W_n(x)$ is independent of both. Therefore, finding a fictitious value of the stiffness $D'(\omega)$ that

$$\omega_{0n}(D',\ M'_w(D'),\ L) = \omega, \qquad M'^2_w(D'(\omega)) = M^2_w D'(\omega)/D, \tag{3.6}$$

we obtain $p_{n2}(\omega) = \delta(M,\ M_w D'(\omega),\ D'(\omega),\ \mu,\ \omega)$.

We will study the behavior of $p_{n2}(\omega)$. When $\omega = \omega_{0n}$ we can directly determine the quantity $p_{n2}(\omega_{0n}) = \delta(M,\ M_w,\ D,\ \mu,\ \omega_{0n})$ from Fig. 1. We will increase ω. Then in solution (3.6) $D'(\omega)$ also increases. In its turn, as the stiffness increases, the curve Ω is displaced to the left since $\omega_{\max}$ (2.1) decreases and, as a result, $p_{n2}(\omega)$ can be determined from the displaced Ω. Similarly, as the frequency decreases, $p_{n2}(\omega)$ can be determined from the curve Ω displaced to the right. As a result, we obtain the dependence $p_{n2}(\omega)$ shown in Fig. 2. In fact, it represents the curve Ω compressed about the vertical straight line $\omega = \omega_{0n}$.

This method does not make it possible to find the quantity $p_{n1}(\omega)$, but since in the high-frequency domain it is of the order of μ and only slightly affects the oscillations, in what follows we will assume that $p_{n1}(\omega) \equiv 0$.

4. BEHAVIOR OF MODES DAMPED IN THE LINEAR APPROXIMATION

Let only the first mode grow in the linear approximation and the remaining modes be damped. We will show that in the nonlinear case we can assume that their amplitudes are small as compared with the amplitude of the first mode. We group the terms with different powers of A_n in (3.2) and take (3.5) into account

$$\frac{\partial^2 A_n}{\partial t^2} - 2p_{n2}(\omega)\frac{\partial A_n}{\partial t} + \Bigg(\omega_{0n}^2 + K\sum_{\substack{m,\ k=1\\ m,\ k\neq n}}^{\infty}(a_{mk}a_{nn} + 2a_{mn}a_{kn})A_mA_k\Bigg)A_n$$
$$+ 3K\Bigg(\sum_{\substack{m=1\\ m\neq n}}^{\infty} a_{mn}a_{nn}A_n\Bigg)A_n^2 + Ka_{nn}^2A_n^3 + K\sum_{\substack{m,\ k,\ j=1\\ m,\ k,\ j\neq n}}^{\infty} a_{mk}a_{jn}A_mA_kA_j = 0. \tag{4.1}$$

The dependence $p_{n2}(\omega)$ should be interpreted as the dependence on the characteristic oscillation frequency.

We will assume that the initial perturbation is small and all the nonlinear terms are also small. Then the square of the characteristic frequency ω is the multiplier of A_n. Since

$$\sum_{\substack{m,\,k=1\\ m,\,k\neq n}}^{\infty}(a_{mk}a_{nn} + 2a_{mn}a_{kn})A_mA_k = \frac{a_{nn}}{\sqrt{2L}}\int_{-L/2}^{L/2}\left(\sum_{\substack{j=1\\ j\neq n}}^{\infty}\frac{\partial W_j(x)}{\partial x}A_j(t)\right)^2 dx$$

$$+ \frac{1}{L}\left(\sum_{\substack{j=1\\ j\neq n}}^{\infty}\int_{-L/2}^{L/2}\frac{\partial W_j(x)}{\partial x}\frac{\partial W_n(x)}{\partial x}A_j(t)\,dx\right)^2 > 0,$$

we have $\omega > \omega_{0n}$. Since the nth mode is linearly stable, $\omega > \omega_{0n} > \omega^*$ and $p_{n2}(\omega) < 0$. Then the term with the first derivative in (4.1) can be interpreted as the action of the artificial internal viscosity in the plate material.

In what follows, we will show that under the action of the gas flow the first oscillation mode is similar to a harmonic oscillation. But from the standpoint of Eqs. (4.1), for the higher modes it is of no importance what is the reason for these oscillations; for example, we may assume that forced oscillations at a given frequency take place in relation to the first mode. Thus, the question of the behavior of the higher modes reduces to the problem of the natural nonlinear oscillations of the plate. This problem has been investigated in detail in the plane [6] and three-dimensional formulations [7], where for the characteristic values of the parameters it was shown that the nonlinear oscillation modes are close to linear for amplitudes not higher than $1.5h$. In terms of Eqs. (4.1) this means that the solutions $|A_n(t)| \ll |A_1(t)|$. For amplitudes much higher than that mentioned above internal parametric resonance may develop and the oscillation modes may change.

5. BEHAVIOR OF THE MODE GROWING IN THE LINEAR APPROXIMATION

Assuming that $|A_n| \ll |A_1|$, $n > 1$, in the equation for A_1 we can omit all the terms containing amplitudes with an index greater than one. Then we can write Eq. (4.1) in the form:

$$\frac{\partial^2 A_1}{\partial t^2} - 2p_{12}(\omega)\frac{\partial A_1}{\partial t} + \omega_{01}^2 A_1 + Ka_{11}^2 A_1^3 = 0. \tag{5.1}$$

We will find the limit cycles of this equation. Since they are periodic solutions, they can be found in the form:

$$A_1(t) = C\cos\omega t + \sum_{n=2}^{\infty}(C_{n1}\cos n\omega t + C_{n2}\sin n\omega t). \tag{5.2}$$

We will assume that the oscillation is close to harmonic, i.e. $|C| \gg |C_{ng}|$. After substituting in (5.1) and retaining only the terms linear in C_{ng}, we obtain

$$-\omega^2 C\cos\omega t - \sum_{n=2}^{\infty} n^2\omega^2(C_{n1}\cos n\omega t + C_{n2}\sin n\omega t) + 2p_{12}(\omega)\omega C\sin\omega t + \omega_{01}^2 C\cos\omega t$$

$$+ \omega_{01}^2\sum_{n=2}^{\infty}(C_{n1}\cos n\omega t + C_{n2}\sin n\omega t) - p_{11}(\omega)C\cos\omega t + Ka_{11}^2C^3\left(\frac{3\cos\omega t}{4} + \frac{\cos 3\omega t}{4}\right) = 0.$$

Equating the coefficients of sines and cosines, we obtain the solution in the first approximation

$$A_1(t) = C\cos\omega t + C_{31}\cos 3\omega t,$$

$$C = \sqrt{\frac{4(\omega^2 - \omega_{01}^2)}{3Ka_{11}^2}}, \quad C_{31} = \frac{Ka_{11}^2}{4(9\omega^2 - \omega_{01}^2)}C^3, \tag{5.3}$$

$$p_{12}(\omega) = 0 \quad \Longrightarrow \quad \omega = \omega', \quad \omega = \omega''.$$

Here, ω' and ω'' (Fig. 2) are, respectively, the solutions of the equations

$$\omega = \omega^*(D'(\omega)), \qquad \omega = \omega^{**}(D'(\omega)), \tag{5.4}$$

where $D'(\omega)$ is the fictitious stiffness (3.6).

Since when the first mode is growing in the first approximation we have $\omega'' < \omega_{01} < \omega'$, from (5.3) we find that oscillations at the frequency ω'' are impossible and there is only a single limit cycle having the frequency ω'.

We will consider the physical meaning of the results obtained. Let us imagine that a small initial perturbation is excited in the plate in accordance with the lowest oscillation mode. The corresponding eigenfrequency lies on the curve Ω on the interval $\omega^{**} < \mathrm{Re}\omega < \omega^*$. In the linear approximation the instability of the perturbation leads to an increase in its amplitude. However, owing to the nonlinear term in (5.1) the amplitude and the frequency are interrelated and the frequency begins to increase, following the amplitude. The increase in frequency signifies motion to the right along the curve $p_{n2}(\omega)$ (Fig. 2). Since without allowance for the action of the gas the solutions of Eq. (5.1) take the form of neutral oscillations with an arbitrary amplitude, the nonlinearity cannot itself lead to cessation of the increase in amplitude. As a result, as long as $p_{n2}(\omega) > 0$ the increase in amplitude, and, consequently, frequency will continue. When the frequency ω reaches ω', the quantity $p_{n2}(\omega)$ becomes equal to zero and amplification gives place to neutral oscillations. In this case no internal resonance takes place since $\omega' < \omega^* < \omega_{0n}$, $n > 1$.

These considerations can be confirmed by energy estimates. We multiply (5.1) by $\partial A_1/\partial t$ and reduce it to the form:

$$\frac{1}{2}\frac{\partial}{\partial t}\left(\left(\frac{\partial A_1}{\partial t}\right)^2 + \omega_{01}^2 A_1^2 + K a_{11}^2 \frac{A_1^4}{2}\right) = 2p_{12}(\omega)\left(\frac{\partial A_1}{\partial t}\right)^2. \tag{5.5}$$

The lefthand side of this equality represents the variation of the total oscillation energy. When $p_{n2}(\omega) > 0$ it increases; when $p_{n2}(\omega) < 0$ it decreases and only when $p_{n2}(\omega) = 0$ are the oscillations neutral. From Eq. (5.5) it directly follows that the limit cycle at the frequency ω' obtained is stable since an increase (decrease) in amplitude leads to an increase (decrease) in frequency and to the opposite pressure effect, i.e., a decrease (increase) in energy and amplitude.

We will find the limits of applicability of formulas (5.3). For this purpose we will calculate the amplitudes C and C_{31} in explicit form. For simplicity, we will consider the case of hinged support on both edges and the absence of tension ($M_w = 0$). We then have

$$\omega_{01} = \sqrt{D}\left(\frac{\pi}{L}\right)^2 \quad \Longrightarrow \quad D'(\omega) = \omega^2\left(\frac{L}{\pi}\right)^4.$$

Since

$$\omega^*(D) \approx \omega_{\max} = (M - 1)^2/\sqrt{D},$$

solving the equation $\omega = \omega^*(D'(\omega))$, we finally obtain the oscillation frequency of the limit cycle: $\omega' = (M - 1)(\pi/L)$. Since at the Mach number M* the system is on the stability limit and $\omega_{01} = \omega^* = \omega'$, we have $\omega_{01} = (M^* - 1)(\pi/L)$. Then from (5.3) we obtain

$$C = \frac{2\pi\sqrt{M + M^* - 2}}{\sqrt{3Ka_{11}^2}L}\sqrt{M - M^*}, \quad a_{11} = \frac{\pi^2}{\sqrt{2}L^{5/2}},$$
$$C_{31} = \frac{2\pi(M + M^* - 2)^{3/2}}{\sqrt{27Ka_{11}^2}L(9(M - 1)^2 - (M^* - 1)^2)}(M - M^*)^{3/2}. \tag{5.6}$$

By virtue of the condition of normalization of the natural modes $W_1(x) = \sqrt{2/L}\cos(\pi x/L)$; therefore, in order to obtain the "physical" amplitude (divided by the thickness) the quantities (5.6) must be multiplied by $\sqrt{2/L}$. In Fig. 3 we have plotted graphs of these dependences for parameters (1.3) and various L. Clearly, the condition $C_{31} \ll C$ is satisfied over the entire range of Mach numbers of practical interest, i.e., the oscillations are close to harmonic.

We note that the frequency dependence of amplitudes (5.3) is the same as for the nonlinear plate oscillations in a vacuum. The difference resides in the fact that in a vacuum the oscillations may takes place at an arbitrary frequency (and amplitude), while in the flow the frequency is determined by the condition $\omega = \omega'$.

6. BEHAVIOR OF THE PLATE AS M INCREASES

Formulas (5.3) and (5.4) show that the oscillation amplitudes increase monotonically with the Mach number M. When M reaches the value M^{**}, the frequency ω_{01} becomes equal to ω^{**} and then leaves the instability domain: $\omega_{01} < \omega^{**}$. At the same time, the oscillations described by the limit cycle obtained in the previous section go on will continuously since the cycle is stable. The two stable solutions, namely, the limit cycle and the undisturbed state, are separated by the unstable limit cycle at the frequency ω'' while appears when $M > M^{**}$. Its instability can be seen from (5.5): the oscillation is damped for an arbitrarily small decrease in amplitude and develops to the limit cycle at the frequency ω' for an arbitrarily small increase.

Thus, we obtain the pattern of the solution attraction domains shown in Fig. 4. This pattern makes it possible to draw a conclusion about the type of excitation of flutter vibrations. As the Mach number increases from 1 to M^* and beyond, flutter is excited softly: the amplitude of the limit cycle is equal to zero when $M = M^{**}$ and gradually increases with M. This is shown, in particular, by formulas (5.6). We increase the Mach number to $M > M^{**}$ and suppress the oscillations, bringing the plate into the stable equilibrium state. Now, we begin to decrease the Mach number. When it becomes equal to M^{**}, the flutter excitation is hard: the amplitude increases sharply from zero to the amplitude of the stable limit cycle. Therein lies the difference between the behavior of the plate as the Mach number increases and decreases.

7. OSCILLATIONS OF RECTANGULAR PLATES

Let us examine the changes in the above analysis when we consider the nonlinear oscillations of a rectangular plate exposed to a supersonic gas flow parallel to one of its edges. We will take the x axis parallel to the gas velocity vector and the y axis perpendicular to it in the plane of the plate. As before, we will use the Kármán model [5, § 7]. The dimensionless equations of motion of an isotropically stretched plate consist of two equations for the deflection w and the Airy stress function Φ:

$$D\Delta^2 w - M_w^2 \Delta w + \frac{\partial^2 w}{\partial t^2} - P\{w\} = \Lambda(w, \Phi),$$
$$\frac{1}{12(1-\nu^2)D}\Delta^2\Phi = -\frac{1}{2}\Lambda(w, w). \tag{7.1}$$

Here, Δ is the two-dimensional Laplace operator, all the dimensionless parameters are the same as in Section 1, and

$$\Lambda(w, \Phi) = \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 w}{\partial y^2}\frac{\partial^2 \Phi}{\partial x^2} - 2\frac{\partial^2 w}{\partial x \partial y}\frac{\partial^2 \Phi}{\partial x \partial y}.$$

The plate occupies the domain $-L_x/2 \le x \le L_x/2$, $-L_y/2 \le y \le L_y/2$. Hinged or clamped boundary conditions jamming are imposed on the deflection w at the edges. In constructing approximate solutions for Φ, it is customary to use integral boundary conditions relating the displacements of the plate edges with the stiffness of the ribs of the structure into which the plate is built.

We will first consider the changes associated with the pressure $P\{w\}$. Considering the deflection in the form of the natural oscillation mode and assuming that the time dependence is harmonic, we will use the pressure calculation method proposed in [3]. The pressure can be calculated in the same way as in the plane case (Section 3) but the Mach number M must be replaced by $M\cos\alpha$, where α is the angle between the plane of the traveling waves of which the natural oscillation consists and the x axis [3]. In particular, for

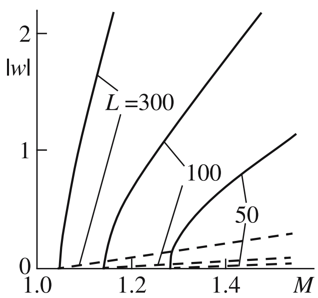

Fig. 3

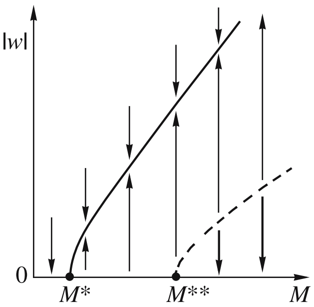

Fig. 4

Fig. 3. Plate deflection amplitudes divided by the thickness (expressions (5.6) multiplied by $\sqrt{2/L}$) for parameters (1.3) and $L = 50$, 100, and 300. The continuous and broken curves correspond to $C(M)$ and $C_{31}(M)$, respectively.

Fig. 4. Domains of attraction of solutions (5.1). The continuous and broken curves correspond to the stable and unstable limit cycles, respectively.

hinged support on all the edges $\alpha = \arctan(L_x n/(L_y m))$, where m and n are the numbers of half-waves of the eigenfunction in the x and y directions, respectively.

We now go over to changes related with the nonlinear terms. To be specific, let there be a high-frequency flutter on the first oscillation mode and, in accordance with the Bubnov–Galerkin method, let us $w(x, y, t) = W_1(x, y)A_1(t)$, where $W_1(x, y)$ satisfies the normalization condition

$$\int\int W_1^2\,dx\,dy = 1,$$

the integration being carried out over the entire surface of the plate. By means of the method described in [5, §19–20], system (7.1) can be reduced to the following equation for the amplitude:

$$\frac{\partial^2 A_1}{\partial t^2} - 2p_{12}(\omega)\frac{\partial A_1}{\partial t} + \omega_{01}^2 A_1 + K_1 A_1^3 = 0.$$

In particular, for hinged support on all the edges and the fixed edge condition (absolutely rigid ribs of the structure in which the plate is embedded) the expression for K_1 has the form:

$$K_1 = D\pi^4 \frac{3(1-\nu^2)(L_x^4 + L_y^4) + 6(L_x^4 + 2\nu L_x^2 L_y^2 + L_y^4)}{L_x^5 L_y^5}.$$

The equation obtained completely coincides with (5.1) if we make the substitutions $Ka_{11}^2 \leftrightarrow K_1$ and $\mathrm{M} \leftrightarrow \mathrm{M}\cos\alpha$. This makes it possible to use the results of Section 5 directly. In the zeroth approximation the stable limit cycle has the form:

$$A_1(t) = C(\omega)\cos(\omega t), \quad C(\omega) = \sqrt{\frac{4(\omega^2 - \omega_{01}^2)}{3K_1}}, \quad \omega = \omega'(M),$$

where $\omega'(M)$ is a solution of the equation $p_{12}(\omega) = 0$ such that $\omega' > \omega_{01}$.

As an example, we will calculate the amplitude of the limit cycle of the oscillations of a hinged plate when $M_w = 0$ assuming that flutter develops in the lowest mode ($m = n = 1$). Since

$$\omega_{01} = \sqrt{D}\,\frac{\pi^2(L_x^2 + L_y^2)}{L_x^2 L_y^2},$$

repeating the algebra of Section 5, we find

$$\omega'(M) = (M\cos\alpha - 1)\frac{\pi\sqrt{L_x^2 + L_y^2}}{L_x L_y}.$$

Since $\omega_{01} = \omega'(M^*)$, we finally obtain

$$C(M) = \frac{2\pi\sqrt{L_x^2 + L_y^2}\sqrt{(M + M^*)\cos\alpha - 2}}{\sqrt{3K_1}L_x L_y}\sqrt{(M - M^*)\cos\alpha}. \tag{7.2}$$

We will consider a $220 \times 750 \times 1.5$ mm steel plate with the wide side positioned across the flow. We will use parameters (1.3), $L_x = 146.67$, and $L_y = 500$. Calculations based on [3] give $M^* = 1.15$, the lowest mode, for which $\alpha = 0.29$, being first to enter the instability domain. Since

$$W_1(x, y) = \frac{2}{\sqrt{L_x L_y}}\cos\left(\frac{\pi x}{L_x}\right)\cos\left(\frac{\pi y}{L_y}\right),$$

it is necessary to multiply expression (7.2) by $2/\sqrt{L_x L_y}$ to obtain the amplitude divided by the plate thickness. In Fig. 5 we have reproduced the calculation results. Clearly, with penetration into the flutter domain the amplitude growth rate is of the same order as in the plane case (Fig. 3). The next oscillation mode (it has two half-waves in the direction of flow and one in the transverse direction) becomes unstable when $M = 1.21$; at this Mach number the amplitude of the lowest mode is approximately equal to the plate thickness.

8. COMPARISON OF THE AMPLITUDES FOR HIGH- AND LOW-FREQUENCY FLUTTER

In the notation used in the present study, for low-frequency flutter the dependence of the oscillation amplitude divided by the thickness on the Mach number for a hinged rectangular plate, calculated in accordance with [4, §4.15–4.18], has the form:

$$A(M) = A_0\sqrt{\frac{M}{M_{\text{cr}} - 1}}, \qquad A_0 = \frac{4}{3}\sqrt{\frac{6L_x^2 + 15L_y^2}{12L_x^2 + (45 - 15\nu^2)L_y^2}}. \tag{8.1}$$

Here, M_{cr} is the critical Mach number for low-frequency flutter

$$M_{\text{cr}} = \frac{D}{\mu L_x^3}\frac{9\pi^4}{16}\left(5 + 2\frac{L_x^2}{L_y^2}\right).$$

These formulas were obtained using the linear piston theory for the gas pressure and the two-term Bubnov–Galerkin approximation. Due to the use of only two terms the value of M_{cr} for plates elongated in the direction perpendicular to flow is underestimated by approximately 20% as compared with the exact value [8]. Nevertheless, formula (8.1) is perfectly suitable for estimating the amplitudes developing in flutter.

For the plate considered in the example in Section 7 the critical Mach number for low-frequency flutter $M_{\text{cr}} = 15.3$. This lies far beyond the limits of applicability of the piston theory. Therefore, in our calculations we will consider plates with the same relative dimensions ($L_x/L_y = 220/750$) but having lower M_{cr} due to their mechanical properties. In Fig. 6 we have reproduced the dependence $A(M)$ for several cases.

Comparing the amplitude growth with penetration into the instability domain for low- (Fig. 6) and high-frequency (Figs. 3 and 5) flutter, we can readily see that the amplitude grows much more rapidly for high-frequency flutter. This is related with the mechanism of maintenance of the nonlinear oscillations

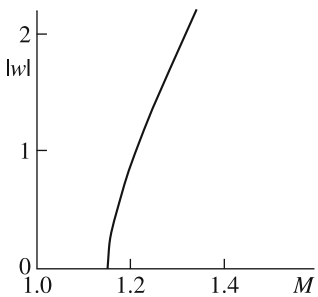

Fig. 5

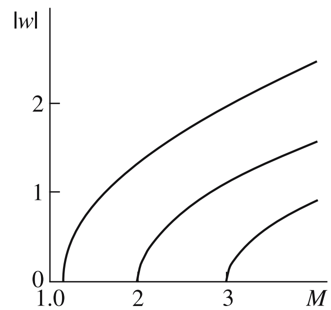

Fig. 6

Fig. 5. Deflection amplitudes of a rectangular plate divided by the thickness (expressions (7.2) multiplied by $2/\sqrt{L_x L_y}$) for parameters (1.3), $L_x = 146.67$, and $L_y = 500$.

Fig. 6. Deflection amplitudes of a rectangular plate divided by the thickness for low-frequency flutter; $L_x/L_y = 146.67/500$, and $M_{\rm cr} = 1.15$, 2, and 3.

in high-frequency flutter: the gas flow "chooses" the oscillation frequency $\omega'(M)$ which strongly depends on the Mach number. In its turn, owing to the nonlinearity, this frequency uniquely determines the amplitude which, as a result, grows strongly with increase in M.

Summary. A new method for calculating the pressure acting on a plate exposed to high-frequency oscillations is proposed. The amplitudes of the nonlinear oscillations developing for in high-frequency plate flutter in the two-dimensional (strip) and three-dimensional (rectangular plate) formulations are investigated using this method.

In the case of instability with respect to a single mode the frequency of the stable limit cycle is determined by the condition of vanishing of the work done by the gas pressure over the oscillation period. The amplitudes of the growing and damped modes are calculated from the frequency using the ordinary equations of nonlinear oscillation of a plate in a vacuum, i.e., the high-frequency flutter oscillations of a plate in a flow can be represented as forced oscillations in a vacuum with a given frequency of the growing mode.

An explicit dependence of the amplitude and the frequency of the limit cycle on the problem parameters is obtained. It is shown that a second limit cycle develops as the Mach number increases. This cycle is unstable and separates the domains of attraction of the state of rest and the stable limit cycle.

The amplitude growth rates with penetration into the instability domain are compared for high- and low-frequency flutters and it is shown that the growth is much stronger for high-frequency flutter.

The work was carried out with support from the Russian Foundation for Basic Research (project No. 05-01-00219) and the Program in Support of Leading Science Schools (NSh-4710.2006.1).

REFERENCES

1. V.V. Vedeneev, "Flutter of a Wide Strip Plate in a Supersonic Gas Flow," Fluid Dynamics **40** (5), 805–817 (2005).
2. V.V. Vedeneev, "High-Frequency Plate Flutter," Fluid Dynamics **41** (2), 313–321 (2006).
3. V.V. Vedeneev, "High-Frequency Flutter of a Rectangular Plate," Fluid Dynamics **41** (4), 641–648 (2006).
4. V.V. Bolotin, *Nonconservative Problems of Elastic Stability Theory* (Fizmatgiz, Moscow, 1961) [in Russian].
5. A.S. Vol'mir, *Nonlinear Dynamics of Plates and Shells* (Nauka, Moscow, 1972) [in Russian].
6. P. Ribeiro and M. Petyt, "Non-Linear Vibration of Beams with Internal Resonance by the Hierarchical Finite Element Method," J. Sound and Vibrat. **224**, No. 4, 591–624 (1999).
7. W. Han and M. Petyt, "Geometrically Nonlinear Vibration Analysis of Thin, Rectangular Plates Using the Hierarchical Finite Element Method. P. I: The Fundamental Mode of Isotropic Plates," Comput. and Structures **63**, No. 2, 295–308 (1997).
8. A.A. Movchan, "Stability of a Panel Moving in a Gas," Prikl. Mat. Mekh. **21**, No. 2, 231–243 (1957).