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Study of the Single-Mode Flutter of a Rectangular Plate in the Case of Variable Amplification of the Eigenmode along the Plate

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Abstract—In [1] the single-mode (high-frequency) flutter of a plate was investigated in the case in which the eigenmodes in a vacuum are associated with symmetric trajectories of disturbance propagation along the plate. In the case of nonsymmetric trajectories a handicap to such a study is presented by the fact that the flow-induced oscillation buildup turns out to be different at different points on the plate, so that the "smearing" of the eigenfunction under the action of elasticity forces must be taken into account. In this study, this case is considered in detail.

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1. FORMULATION OF THE PROBLEM

A thin, elastic, isotropically extended rectangular plate is on one side in a uniform supersonic inviscid perfect-gas flow, while on its other side a constant pressure, equal to the undisturbed gas pressure, is maintained. The plate is mounted in an absolutely rigid surface separating the flow from the constant-pressure region. The system stability is investigated under the assumption that the gas effect on plate oscillations is small.

We will introduce a coordinate system xyz, as shown in Fig. 1. The gas flows in the region $z > 0$ by an angle θ to the x axis, while the plate occupies the region W: $|x| < L_x/2$, $|y| < L_y/2$, $z = 0$. The plate width L_x and length L_y are non-dimensionalized by its thickness. Apart from these, the problem involves the following dimensionless parameters

$$\mathrm{M} = \frac{u}{a}, \qquad \mathrm{M}_w = \frac{\sqrt{\sigma/\rho_m}}{a}, \qquad D = \frac{D_w}{a^2\rho_m h^3}, \qquad \mu = \frac{\rho}{\rho_m}.$$

Here, u, a, and ρ are the flow velocity, the speed of sound, and the gas density, σ is the extending stress in the middle plane of the plate, ρ_m and h are the plate material density and the plate thickness, and $D_w = Eh^3/(12(1 - \nu^2))$ is its flexural rigidity. We will assume that $\mathrm{M} > 1$ and $\mu \ll 1$. We will also assume that $L_x \gg 1$ and $L_y \gg 1$. The differential equations and the boundary conditions are presented in [1].

2. RESULTS OF PREVIOUS STUDIES

We will present the main results of the study [1]. We will consider the shape of plate oscillations in a vacuum, which is characterized by the numbers m and n of half-waves in the direction x and y, respectively. Far away from the plate edges it can be represented in the form (Fig. 2a):

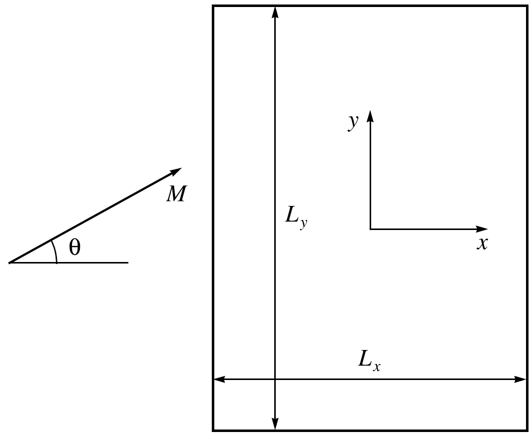

Fig. 1. Plate in a gas flow.

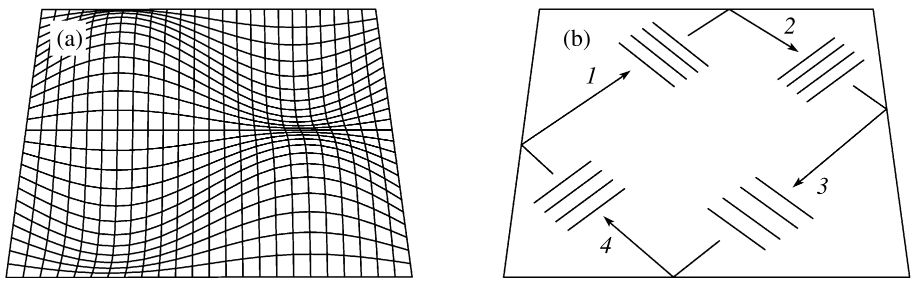

Fig. 2. Eigenmode of the plate in a vacuum (a) and its representation as the superposition of traveling waves (b).

$$w(x,\ y,\ t) = \cos(k_x x + \varphi_x)\cos(k_y y + \varphi_y).e^{-i\omega_0 t}. \tag{2.1}$$

Here, w is the plate deflection, $\omega_0 \in \mathbf{R}$ is the oscillation frequency, and k_x, k_y, φ_x, and φ_y are the wavenumbers and the phase shifts dependent on the half-wave numbers m and n, the boundary conditions, and the properties and the dimensions of the plate. The wavenumbers are related with the frequency by the dispersion equation

$$Dk_0^4 + \mathrm{M}_w^2 k_0^2 - \omega_0^2, \qquad k_0 = \sqrt{k_x^2 + k_y^2}. \tag{2.2}$$

We will represent the oscillation shape (2.1) in the form of the superposition of four traveling waves

$$w(x,\ y,\ t) = \frac{1}{4}(e^{i\varphi_x}e^{ik_x x} + e^{-i\varphi_x}e^{-ik_x x})(e^{i\varphi_y}e^{ik_y y} + e^{-i\varphi_y}e^{-ik_y y})e^{-i\omega_0 t} = C_1 e^{i(k_x x + k_y y - \omega_0 t)} + C_2 e^{i(k_x x - k_y y - \omega_0 t)} + C_3 e^{i(-k_x x - k_y y - \omega_0 t)} + C_4 e^{i(-k_x x + k_y y - \omega_0 t)}. \tag{2.3}$$

We will number the wave propagation directions in accordance with the numbers of the terms in Eq. (2.3) (Fig. 2b). Then the formation of a standing wave can be represented as follows. A traveling wave is excited on one of the plate edges and propagates, for example, in direction *1*. Being successively reflected from the four plate edges, it transforms in waves traveling in directions *2*, *3*, and *4*. On the last reflection, it transforms to the original wave, whereupon the process is cyclically repeated. After several cycles the motion of the four above-noted waves becomes steady and their superposition leads to the formation of a standing wave.

Let now the plate be in a gas flow. We will neglect the effect of the plate edges on the gas flow disturbance and will assume that the flow acts on the traveling waves (2.3) as though they were infinite in extent. The

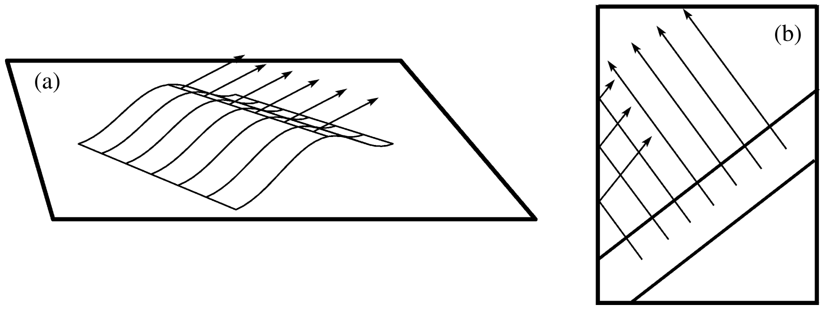

Fig. 3. Motion of a wave as motion of its individual parts.

flow action on plane waves was studied in [2, 3]; it is also described in [1]. The dispersion equation of the plate/gas system is as follows:

$$(Dk^4 + \mathrm{M}_w^2 k^2 - \omega^2) - \mu \frac{(\omega - \mathrm{M}k\cos\alpha_j)^2}{\sqrt{k^2 - (\omega - \mathrm{M}k\cos\alpha_j)^2}} = 0. \quad (2.4)$$

Here, α_j is the angle between the flow velocity vector and the wave vector, $j = 1, \ldots, 4$. Considering μ as a small parameter, letting $\omega = \omega_0 + \Omega_{\mathrm{loc}}$, $|\Omega_{\mathrm{loc}}| \ll \omega_0$, and solving Eq. (2.4) for k we obtain

$$k = k_0 + \Delta(k_0) + g^{-1}(\omega_0)\Omega_{\mathrm{loc}}, \quad (2.5)$$

where $\Delta(k_0)$ is the leading term of the expansion of $k(\mu)$ for small μ and $g(\omega_0) = \partial\omega_0/\partial k_0$ is the group velocity of the waves, where the derivative is taken with account of Eq. (2.2).

We will consider the action of the gas on a natural oscillation representing it as a sequence of traveling wave reflections, as described above. For this purpose, the wave motion can conveniently be represented as the motion of its individual regions (Fig. 3). Between the edges, the trajectories of these regions are segments of straight lines having one of the four directions (Fig. 2b), with mirror reflection from the edges. Depending on the oscillation form under consideration, the trajectories may be both closed (Fig. 4, a and b) and nonclosed (Fig. 4c). A closed trajectory is a closed polygonal line. A nonclosed trajectory is everywhere dense within the rectangle outlined by the plate contour.

We will call the time interval, for which the trajectories of the wave parts return to their initial points (in the case of a closed trajectory) or to closely located points (in the case of a nonclosed one), the reflection cycle of the wave parts. Generally, this cycle does not coincide with the cycle of reflections of the waves as a whole, which is shown in Fig. 2a and always consists of four reflections. In fact, after four reflections any wave part can reflect not only in itself but in another part as well, so that reflections of these parts do not constitute a closed cycle.

Let us consider any trajectory and evaluate the amplitude variation for a reflection cycle. It occurs, firstly, during the wave motion from one edge to another due to the presence of the imaginary part of the wavenumber (2.5) (motion along the trajectory links, the non-zero imaginary part is due to the gas action on the wave) and, secondly, on reflections from the plate edges. Letting the initial amplitude to be equal to unity and considering successively the amplitude variations in the motion between the edges and on reflections we obtain the amplitude after a reflection cycle (it is assumed that n reflections take place for the time interval under consideration)

$$\prod_{p=1}^{n} A_p \exp(-l_1 \operatorname{Im}\Delta(k_1) - l_2 \operatorname{Im}\Delta(k_2) - l_3 \operatorname{Im}\Delta(k_3) - l_4 \operatorname{Im}\Delta(k_4)).$$

Here, A_i are the coefficients of reflection from the edges, l_j are the total distances passed by the trajectory along direction j, and $\Delta(k_j)$ are additions to the wavenumbers due to the flow effect on the waves traveling

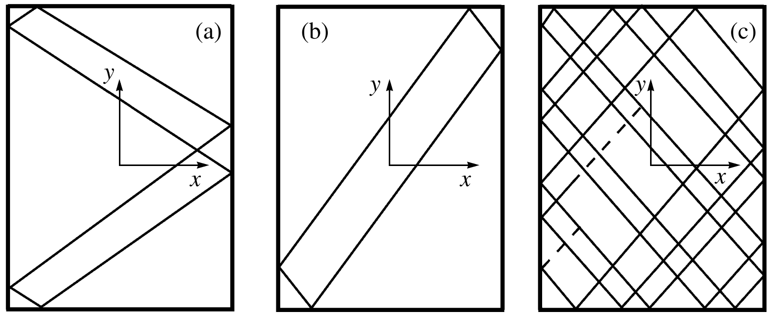

Fig. 4. Trajectories of the motion of parts of the wave, closed and symmetric about one of the axes (a), closed and nonsymmetric about the axes (b), and unclosed (c).

along the plate [1]. Since in the absence of the gas the amplitude remains invariant after a reflection cycle (since the plate in itself is a conservative system), we have

$$\prod_{p=1}^{n} A_p = 1.$$

Then, obviously, $l_1 = l_3$ and $l_2 = l_4$ (in the case of a nonclosed trajectory these equalities are only approximate). Because of this, after a reflection cycle the amplitude becomes as follows:

$$\exp(-l_1\,\mathrm{Im}(\Delta(k_1) + \Delta(k_3)) - l_2\,\mathrm{Im}(\Delta(k_2) + \Delta(k_4))). \tag{2.6}$$

If the exponent is positive, then after each reflection cycle the oscillation amplitude increases, that is, we deal with a growing mode. If the exponent is negative, then the mode is decaying. This quantity is related with the temporal growth rate on a given trajectory Ω_{loc} by the equation [4]

$$\mathrm{Im}\,\Omega_{\mathrm{loc}} = -g(\omega_0)\frac{l_1\,\mathrm{Im}(\Delta(k_1) + \Delta(k_3)) + l_2\,\mathrm{Im}(\Delta(k_2) + \Delta(k_4))}{2(l_1 + l_2)}. \tag{2.7}$$

The sign of $\mathrm{Im}\,\Omega_{\mathrm{loc}}$ makes it possible to determine the presence or the absence of the amplitude growth for any trajectory of the given oscillation form. If the amplitude variation is the same on different trajectories (that is, $\mathrm{Im}\,\Omega_{\mathrm{loc}}$ is the same), then a plane wave remains plane, the oscillation form does not change, and $\omega = \omega_0 + \Omega_{\mathrm{loc}}$ is the natural frequency of the plate oscillations in the flow. In this case, the stability criterion is given by the condition

$$\mathrm{Im}\,\Omega_{\mathrm{loc}} < 0. \tag{2.8}$$

This case was completely studied in [1].

If, however, $\mathrm{Im}\,\Omega_{\mathrm{loc}}$ is different on different trajectories, then the amplitude changes in a different way on the neighboring wave parts, a plane wave ceases to be plane, and the above reasoning does not hold. In this case, apart from the motion of wave parts along trajectories, the eigenfunction "smears" in the direction perpendicular to the trajectories. The eigenfunction generated as a result of this smearing can be strongly different from the original function. Nevertheless, some conclusions on the stability can be made in this case as well: if on all trajectories $\mathrm{Im}\,\Omega_{\mathrm{loc}} > 0$, then the plate oscillations build up and if on all trajectories $\mathrm{Im}\,\Omega_{\mathrm{loc}} < 0$, then they decay. In the case in which $\mathrm{Im}\,\Omega_{\mathrm{loc}}$ changes sign, an apriori conclusion on whether the eigenfunction generated as a result of disturbance diffraction (smearing) transverse to the trajectories is decaying or growing cannot be made. This study is devoted to the investigation of all these cases.

We will preliminarily consider the conditions determining whether the value given by Eq. (2.7) is dependent or independent of a trajectory. We will consider only the case of an "obliquely" incident flow, since in the case of a flow perpendicular to one of the edges we have $\Delta(k_1) = \Delta(k_2)$ and $\Delta(k_3) = \Delta(k_4)$, the quantity $\mathrm{Im}\,\Omega_{\mathrm{loc}}$ is the same on all trajectories of the given eigenmode, and the stability criterion (2.8) performs.

All trajectories can be divided into three groups, namely, closed trajectories symmetric about the x or y axes, closed trajectories nonsymmetric about the x or y axes (in this case they are symmetric about the center of the plate), and nonclosed trajectories (Fig. 4). Here, the case of nonclosed trajectories will not be considered.

Let us consider the case of closed symmetric trajectories, for the sake of definiteness, those symmetric about the x axis. For them we have always $l_1 = l_4$ and $l_2 = l_3$. In view of the fact that for any closed trajectories $l_1 = l_3$ and $l_2 = l_4$, the distances passed by a disturbance are the same in all four directions. Since the total length of a trajectory is independent of the initial point from which the trajectory has proceeded, the disturbance amplification (2.7) is also independent of this point and the oscillation amplitude grows uniformly throughout the entire plate. This case was studied; it was shown that it leads to Eq. (2.8)

We will now consider closed nonsymmetric trajectories. It can readily be seen that in this case the growth rates will be different for trajectories proceeding from different points. The trajectories proceeding from the corner points are extremal; there are two such trajectories connecting the opposite corners of the plate. On one of these, which will be called the "maximum" trajectory, the disturbance grows most rapidly, while on the other it grows most slowly. Hence follows that the plate oscillation amplitude grows nonuniformly throughout the plate, namely, the oscillation builds up most rapidly at points close to the maximum trajectory. The greater the difference between the distances l_1 and l_2 (that is, the less the number of reflections for one cycle) the more clearly this effect is expressed. In particular, it is most clearly expressed in the case in which the maximum trajectory is the diagonal of the rectangle (Fig. 4b); in this case $l_2 = 0$. Thus, in the case of closed nonsymmetric trajectories criterion (2.8) does not hold and it is necessary to consider the mechanism of disturbance smearing in the directions perpendicular to the trajectories.

3. DERIVATION OF THE GOVERNING EQUATION

For the sake of simplicity, we will first consider the case of a square plate with the eigenmode associated with the trajectories whose directions are parallel to the diagonals. We will introduce a $(\xi,\ \eta)$ coordinate system, as shown in Fig. 5. In these coordinates

$$l_1 = 2\big(\eta_{\max} - |\eta|\big), \qquad l_2 = 2|\eta|,$$

where $\eta_{\max}$ is a half-length of the square diagonal. Each trajectory is determined by preassigning one of the coordinates, for example, η. Then we have

$$\operatorname{Im}\Omega_{\text{loc}}(\eta) = -g(\omega_0)\frac{l_1(\eta)\operatorname{Im}(\Delta(k_1) + \Delta(k_3)) + l_2(\eta)\operatorname{Im}(\Delta(k_2) + \Delta(k_4))}{4\eta_{\max}}$$
$$= -\frac{g(\omega_0)}{2}\operatorname{Im}(\Delta(k_1) + \Delta(k_3)) + |\eta|\frac{g(\omega_0)}{2\eta_{\max}}\operatorname{Im}(\Delta(k_1) + \Delta(k_3) - \Delta(k_2) - \Delta(k_4)).$$

We will use the designation

$$\Omega_{\text{loc}}(\eta) = c_1|\eta| + c_2. \tag{3.1}$$

Without loss of generality, we will assume that the diagonal $\eta = 0$ is the maximum trajectory. Then we have

$$\operatorname{Im} c_1 = \frac{g(\omega_0)}{2\eta_{\max}}\operatorname{Im}(\Delta(k_1) + \Delta(k_3) - \Delta(k_2) - \Delta(k_4)) < 0,$$
$$\operatorname{Im} c_2 = -\frac{g(\omega_0)}{2}\operatorname{Im}(\Delta(k_1) + \Delta(k_3)) > 0. \tag{3.2}$$

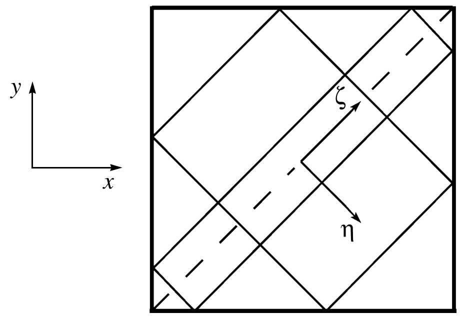

Fig. 5. Trajectories and $(\xi,\ \eta)$ coordinate system in the case of a maximum trajectory connecting the opposite corners of the plate (broken line).

We will consider the equation of plate oscillations in a gas flow

$$D\left(\frac{\partial^4 w}{\partial \xi^4} + 2\frac{\partial^4 w}{\partial \xi^2 \partial \eta^2} + \frac{\partial^4 w}{\partial \eta^4}\right)$$
$$- \mathrm{M}_w^2\left(\frac{\partial^2 w}{\partial \xi^2} + \frac{\partial^2 w}{\partial \eta^2}\right) - \omega^2 w + p(w,\ \omega) = 0.$$

Since, by assumption, the greatest amplification occurs on the plate diagonal, and the wave vector is also aligned with this diagonal, we will assume that the eigenfunction is considerably different from zero only in the vicinity of the maximum trajectory, and will seek the solution in the form:

$$w(\xi,\ \eta,\ t) = \sin(k_0\xi + \varphi)A(\eta)e^{-i\omega t}, \qquad k_0 = \sqrt{k_x^2 + k_y^2}. \tag{3.3}$$

Clearly that at the plate corners this solution satisfies the boundary conditions only in the "integral" fashion rather than exactly. Because of this, we will consider the points on the plate that lie outside certain vicinities of the corners. Substituting Eq. (3.3) in the equation of motion yields

$$\sin(k_0\xi + \varphi)\left(D\frac{d^4A(\eta)}{d\eta^4} - (2Dk_0^2 + \mathrm{M}_w^2)\frac{d^2A(\eta)}{d\eta^2}\right.$$
$$\left. + (Dk_0^4 + \mathrm{M}_w^2)k_0^2 - \omega^2)A(\eta)\right) + p(\sin(k_0\xi + \varphi)A(\eta),\ \omega) = 0. \tag{3.4}$$

We will now make the main assumption. We will assume that the pressure $p(\sin(k_0\xi + \varphi)A(\eta),\ \omega)$ can be so calculated as if $A(\eta) \equiv \mathrm{const}$. Physically, this means that the "smearing" of the natural form occurs only at the expense of the elastic forces generated in the plate rather than due to the flow. We will consider Eq. (3.4) at $A(\eta) \equiv \mathrm{const}$ letting the derivatives with respect to η to be equal to zero and compare the equation thus obtained with the dispersion equation on a trajectory with a given η in which the term with the pressure is averaged over the trajectory

$$Dk_0^4 + \mathrm{M}_w^2 k_0^2 - \omega^2 + \alpha(k_0,\ \omega,\ \eta) = 0. \tag{3.5}$$

Here, $\alpha(k_0,\ \omega,\ \eta)$ represents the corresponding terms of Eq. (2.4) averaged over the trajectory

$$\alpha(k_0,\ \omega,\ \eta) = -\frac{\mu}{l_1 + l_2 + l_3 + l_4}\sum_{j=1,\ldots,\ 4} l_j(\eta)\frac{(\omega - \mathrm{M}k_0\cos\alpha_j)^2}{\sqrt{k_0^2 - (\omega - \mathrm{M}k_0\cos\alpha_j)^2}}.$$

Equation (3.5) may be considered as the averaged dispersion equation along the entire trajectory (of course, the actual dispersion equations (2.4) are their own for any of the four waves). Comparing Eqs. (3.5) and (3.4) term by term we obtain

$$p\sin(k_0\xi + \varphi)A(\eta)A(\eta),\ \omega) = \alpha(k_0,\ \omega,\ \eta)A(\eta)\sin(k_0\xi + \varphi), \tag{3.6}$$

where $\alpha(k_0, \omega, \eta)$ is determined using the "trajectory method" with a trajectory proceeding from the point with the given value of η. We will now relate α with the "local" amplification on the trajectory. Let the "local" frequency on the trajectory $\omega = \omega_0 + \Omega_{\text{loc}}(\eta)$, $|\Omega_{\text{loc}}(\eta)| \ll \omega_0$. Then at a fixed η Eq. (3.5) is transformed as follows:

$$Dk_0^4 + \mathrm{M}_w^2 k_0^2 - (\omega_0 + \Omega_{\text{loc}}(\eta))^2 + \alpha(k_0, \omega_0 + \Omega_{\text{loc}}(\eta), \eta) = 0. \tag{3.7}$$

Using Eq. (2.2) and in view of the smallness of $\Omega_{\text{loc}}(\eta)$ we obtain

$$\alpha(k_0, \omega_0 + \Omega_{\text{loc}}(\eta), \eta) = 2\omega_0 \Omega_{\text{loc}}(\eta). \tag{3.8}$$

We will now return to Eq. (3.4) which takes account for the disturbance smearing at the expense of the elastic forces. Letting the natural frequency to be slightly different from the oscillation frequency in a vacuum, $\omega = \omega_0 + \Omega$, $|\Omega| \ll \omega_0$, and substituting the expression for α in Eq. (3.4) we obtain

$$D\frac{d^4A(\eta)}{d\eta^4} - (2Dk_0^2 + \mathrm{M}_w^2)\frac{d^2A(\eta)}{d\eta^2} - 2\omega_0(\Omega - \Omega_{\text{loc}}(\eta))A(\eta) = 0. \tag{3.9}$$

This is the fundamental eigenvalue problem which will be studied below.

4. SOLUTION OF THE EIGENVALUE PROBLEM

We will consider eigenfunctions with a fairly large number of flexural waves, that is, we will assume that $k_0 \gg 1$. Then we have

$$\varepsilon\frac{d^4A(\eta)}{d\eta^4} - \frac{d^2A(\eta)}{d\eta^2} - q(\Omega - \Omega_{\text{loc}}(\eta))A(\eta) = 0, \tag{4.1}$$

$$\varepsilon = \frac{1}{2Dk_0^2 + \mathrm{M}_w^2} \ll 1, \quad q = \frac{2\omega_0}{2Dk_0^2 + \mathrm{M}_w^2}, \quad 0 < q < \frac{1}{\sqrt{D}}.$$

In accordance with the general notions of the behavior of the solutions of equations with a small coefficient of the higher-order derivative, solutions (4.1) can be subdivided into two types. The solution of the first type tend, as $\varepsilon \to 0$, to the solutions of the limiting equation

$$\frac{d^2A(\eta)}{d\eta^2} + q(\Omega - \Omega_{\text{loc}}(\eta))A(\eta) = 0. \tag{4.2}$$

Since the order of the limiting equation is lower, its solutions satisfy a smaller number of boundary conditions. For this reason, as $\varepsilon \to 0$, in the vicinities of the domain boundaries solutions (4.1) possess a "boundary layer", that is, a transition region in which the solution changes sharply in order for the boundary conditions, which are "superfluous" from the standpoint of Eq. (4.2), could be satisfied.

The solutions of the second type are, as $\varepsilon \to 0$, rapidly varying functions (rapidly oscillating or rapidly decaying) which can be sought for using the WKB method. We will consider a dispersion equation corresponding to Eq. (4.1) by "freezing" the nonconstant coefficient of the last term and substituting the solution of the form $A(\eta) = e^{is\eta}$ in Eq. (4.1). Then we obtain

$$\varepsilon s^4 - s^2 - q(\Omega - \Omega_{\text{loc}}(\eta)) = 0. \tag{4.3}$$

Two solutions of Eq. (4.3) corresponding to the WKB approximation (two other solutions correspond to approximation (4.2)) are as follows:

$$s_{1,2}(\eta) = \frac{1 + \sqrt{1 + 4q\varepsilon(\Omega - \Omega_{\text{loc}}(\eta))}}{2\varepsilon}.$$

Solutions of Eq. (4.1) of the WKB type are as follows:

$$A(\eta) = C_{1,2}(\eta)\exp\left(i\int s_{1,2}(\eta)\,d\eta\right), \tag{4.4}$$

where $C_{1,2}(\eta)$ are some slowly varying functions. Clearly, functions (4.4) decay rapidly with distance from the plate edges. In constructing the eigenfunction in the central region of the plate, that is, outside of a certain vicinity of the edges, the partial solutions (4.4) may be excluded from consideration.

Thus, in what follows we will consider Eq. (4.2). By virtue of Eq. (3.1), it can be rewritten in the form:

$$\frac{d^2A(\eta)}{d\eta^2} + q(\Omega - c_1(\eta) - c_2)A(\eta) = 0. \tag{4.5}$$

Let us consider the region $\eta > 0$ within which $|\eta| = \eta$. We will make the change of the variable $\zeta = q^{1/3}c_1^{-2/3}(c_1\eta + c_2 - \Omega)$ assuming that $-\pi/3 < \arg c_1^{1/3} < \pi/3$. Then Eq. (4.5) can be rewritten in the form:

$$\frac{d^2A(\zeta)}{d\zeta^2} - \zeta A(\zeta) = 0. \tag{4.6}$$

This is the Airy equation. It possesses two independent solutions, one of which, the Airy function $\mathrm{Ai}(\zeta)$, vanishes, as $|\zeta| \to \infty$ in the sector $-\pi/3 < \arg\zeta < \pi/3$ (which corresponds to real $\eta \to +\infty$). The second solution, namely, the Airy function of the second kind $\mathrm{Bi}(\zeta)$, increases without bounds as $|\zeta| \to \infty$ in the above-noted sector and is not appropriate from the meaning of the problem. Thus, for $\eta > 0$ we have

$$A(\eta) = \mathrm{Ai}\left(\left(\frac{q}{c_1^2}\right)^{1/3}(c_1\eta + c_2 - \Omega)\right).$$

To determine the solution for $\eta < 0$, in view of the symmetry of Eq. (4.2), it is necessary to let $A(\eta) = A(-\eta)$. This ensures the continuity of $A(\eta)$ and all its even derivatives at zero. For the continuity of the first and all other odd derivatives (that is, for the existence of solution (4.5) for all η) it is necessary and sufficient that the following condition is fulfilled

$$\left.\frac{dA(\eta)}{d\eta}\right|_{\eta=0} = 0 \Rightarrow \left.\frac{d\mathrm{Ai}(\eta)}{dz}\right|_{z=q^{1/3}c_1^{-2/3}(c_2-\Omega)} = 0.$$

The derivative of the Airy function has a denumerable set of zeros z_N, all of which are real and negative[1]. Thus, we obtain the frequency equation

$$\Omega_N = c_2 - z_N\left(\frac{c_1^2}{q}\right)^{1/3}, \qquad N \in \mathbf{N} \tag{4.7}$$

and the eigenfunctions in the entire domain $\eta \in \mathbf{R}$

$$A_N(\eta) = \mathrm{Ai}\,(q^{1/3}c_1^{1/3}|\eta| + z_N). \tag{4.8}$$

The values of the first two z_N are numerically determined, while the other values can be found using the asymptotic behavior of the Airy function

$$z_1 \approx 1.019, \quad z_2 \approx 3.248, \quad z_N \approx -\left(\frac{3\pi}{8} + \frac{3\pi(n-1)}{2}\right)^{2/3}, \quad N > 2.$$

The eigenvalue with the greatest imaginary part is as follows:

$$\Omega_1 = c_2 + 1.019\left(\frac{c_1^2}{q}\right)^{1/3}.$$

[1] See http://dlmf.nist.gov/9/9/

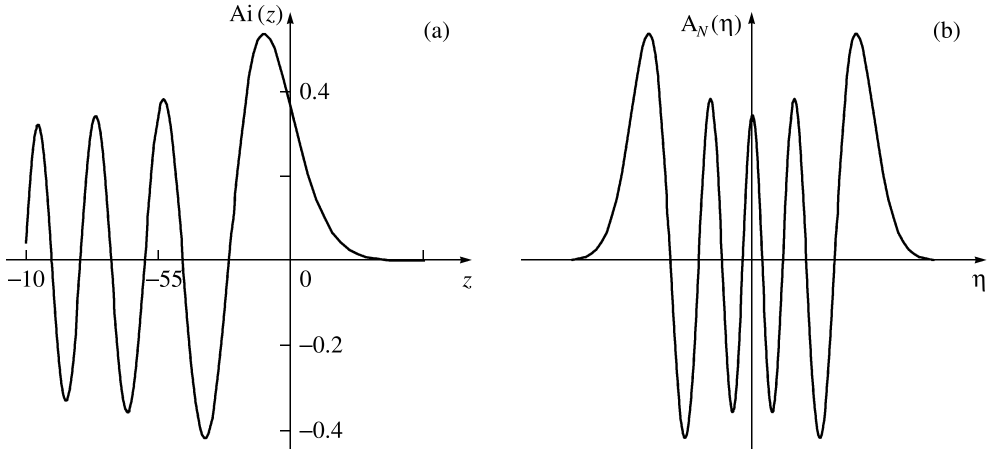

Fig. 6. Airy function of the real argument (a) and eigenmode $A_N(\eta)$ (b).

5. ANALYSIS OF THE SOLUTION

Any eigenmode of the plate oscillations in a vacuum is associated with a denumerable set of eigenmodes of the oscillations in a gas flow.

We will consider the structure of the eigenfunctions (4.8). The Airy function $\mathrm{Ai}(z)$ decays as $z \to +\infty$, $z \in \mathbf{R}$ and oscillates as $z \to -\infty$ (Fig. 6a). In order to obtain a function symmetric about $\eta = 0$ it must be so displaced that the point with a horizontal tangent turns out to be at zero. Then its reflection in the left half-plane united with the original in the right part gives a smooth function for all real η, that is, the eigenfunction (Fig. 6b). The functions $|A_N(\eta)|$ monotonically tend to zero as $\eta \to \pm\infty$, $|\eta| > \eta_N$, and oscillate for $|\eta| < \eta_N$.

The natural frequencies (4.7) have an imaginary part which is always less than the maximum "local" amplification along the maximum trajectory $\Omega_{\mathrm{loc}}(0)$. Thus, the minimum trajectory $\xi = 0$ always "neutralizes" a portion of the amplification occurring in the vicinity of the maximum trajectory. The stability criterion is given by the inequality $\mathrm{Im}\,\Omega_1 < 0$, that is

$$-\frac{g(\omega_0)}{2}\,\mathrm{Im}(\Delta(k_1) + \Delta(k_3)) + 1.019\left(\frac{2Dk_0^2 + \mathrm{M}_w^2}{2\omega_0}\right)^{1/3}$$
$$\times\left(\frac{g(\omega_0)}{2\eta_{\max}}\,\mathrm{Im}(\Delta(k_1) + \Delta(k_3) - \Delta(k_2) - \Delta(k_4))\right)^{2/3} < 0.$$

Among the denumerable set of the eigenmodes (4.8) corresponding to a single natural mode in a vacuum, only a finite number of the modes can be growing. In fact, in view of Eq. (4.7) and the inequality $\mathrm{Im}\,c_1 < 0$, we obtain that, starting from a certain N, the inequality $\mathrm{Im}\,\Omega_N < 0$ is fulfilled.

Since $c_1 \sim \mu/\eta_{\max}$ in the order [1], for the point η_N in which the oscillating behavior changes for the decaying behavior, we can write

$$\eta_N \sim \frac{|z_N|}{(qc_1)^{1/3}} \sim |z_N|\left(\frac{\eta_{\max}}{\mu}\right)^{1/3}.$$

For the correctness of the solution obtained it is necessary that A_N be small in the vicinity of the plate edges. This is fulfilled when $\eta_N \ll \eta_{\max}$. Assuming that $\mu \sim 10^{-4}$ and $z_N \sim 1$ we obtain $\eta_{\max} > 100$, which, even for $N = 1$, is valid only for fairly large $\eta_{\max}$, that is, for large-sized plates. Otherwise, the eigenfunctions obtained will interact with the discarded "boundary" solutions of the WKB type and the structure of the eigenfunctions determined as a result of this interaction, together with their natural frequencies, will be more intricate.

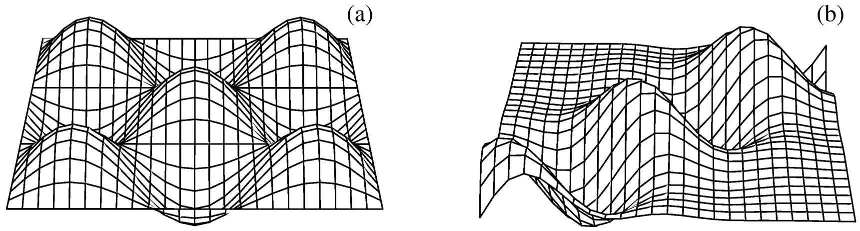

Fig. 7. Eigenmodes of the plate in a vacuum (a) and in a flow (b).

In the case of plates of very large dimensions $\eta_{\max} \to \infty$, $c_1 \to 0$, and frequencies (4.7) are similar in value to the local frequency along the maximum trajectory. Since in this case the width of the region, where the displacement is nonzero, is small as compared with the plate width ($\eta_N \ll \eta_{\max}$), the eigenfunction is concentrated in a narrow band near the maximum trajectory.

We note the following interesting fact. In a vacuum the plate has a denumerable set of the eigenmodes each of which has a form shown in Fig. 7a. Let us consider a mode associated with closed trajectories, nonsymmetric about the x and y axes. Let the plate be in a flow perpendicular to the leading edge. Then the form of deflection in the flow, corresponding to this mode, changes slightly, namely, the wave amplitude increases near the trailing edge, since the wave vectors acquire the complex addition $\Delta(k)$. The position of node lines remains unchanged. The frequency acquires a small complex addition, that is, this mode becomes either growing or decaying.

We will now so rotate the flow velocity vector that the flow past the plate becomes "oblique". Then this eigenmode vanishes and is replaced by a denumerable set of modes concentrated in the vicinity of the maximum trajectory. Each of these has form (3.3) (Fig. 7b). It is interesting to note that for real η the functions $A_N(\eta)$ are complex and eigenmodes (3.3) are not standing waves. The "physical" solutions, that is, the real and imaginary parts (3.3) appear as follows: from the maximum trajectory there diverge traveling waves with the wave vector perpendicular to the trajectory and directed away from it (since $d \arg A_N(\eta)/d\eta > 0$ for $\eta > 0$), while the fixed envelope of these waves oscillates for $|\eta| < |\eta_N|$ and decays for larger η.

In the case in which one of the eigenmodes obtained is the most growing among all the modes of the plate in a flow, the previous analysis gives the stability criterion and the form of the most growing oscillation. If there exists a mode of another family, whose frequency is close to that of one of the modes determined, then, in view of the approximate formulation of the problem (as regards the calculation of the pressure), the exact natural form of the plate deflection can be a combination of the approximate forms. However, the frequency remains the same. Thus, the inequality $\operatorname{Im}\Omega_1 > 0$ remains the sufficient instability condition.

6. GENERAL CASE

Above we considered the case of a square plate and an eigenmode associated with the maximum trajectory in the form of the square diagonal. All the reasoning and results can be transferred unchanged to the case of a rectangular plate and an eigenmode, whose maximum trajectory connects the opposite corners of the plate. For this purpose, it is necessary to introduce a coordinate system, as shown in Fig. 8. It is constant in the vicinity of the links of the maximum trajectory and rotates jumpwise on reflection from the edges. Clearly, the solution constructed in this coordinate system is not valid in the vicinity of the reflection points and the plate corners. If the plate dimensions are sufficiently large, then the dimensions of these vicinities are small, as compared with the link lengths, and the solution will be valid "almost" everywhere.

In this coordinate system an expression of the form (3.1) can be written, whereupon all the calculations leading to Eqs. (3.6) and (3.9) and, ultimately, to solution (4.7), (4.8) can be verbatim repeated. The only difference is that the constants c_1 and c_2 will be determined by a particular oscillation mode and, generally, will be different from (3.2).

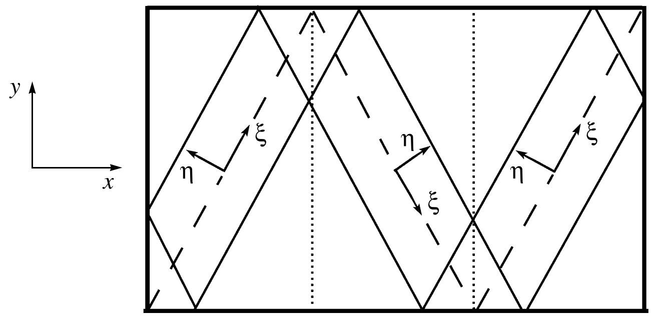

Fig. 8. Trajectories and (ξ, η) coordinate system in the general case of a closed nonsymmetric trajectory (broken line).

Summary. The eigenmodes of plate oscillations in a supersonic flow are studied in the case in which the oscillation amplification produced by the flow is different at different points of the plate, so that the eigenmode "smearing" effect must be taken into account.

Any such eigenmode of the plate in a vacuum is associated with a denumerable set of eigenmodes of the plate in a flow. They are nonzero in the vicinity of the polygonal line connecting the opposite corners of the plate and vanish at a fairly large distance from it. The natural frequencies are determined and the stability criterion is derived. The growth rate of the most growing oscillation is always less than the maximum local growth rate determined using the method of "trajectories".

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REFERENCES

1. V.V. Vedeneev, "High-Frequency Flutter of a Rectangular Plate," Fluid Dynamics **41** (4), 641 (2006).
2. V.V. Vedeneev, "Flutter of a Wide Strip Plate in a Supersonic Gas Flow," Fluid Dynamics **40** (5), 805 (2005).
3. V.V. Vedeneev, "High-Frequency Plate Flutter," Fluid Dynamics **41** (2), 313 (2006).
4. A.G. Kulikovskii, "Global Instability of Homogeneous Flows in Non-One-Dimensional Domains," Prikl. Mat. Mekh. **70**, 257 (2006).
5. A.G. Kulikovskii, "Stability of Homogeneous States," Prikl. Mat. Mekh. **30**, 148 (1966).