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Single-Mode Plate Flutter Taking the Boundary Layer into Account

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Abstract—The stability of an elastic plate in supersonic gas flow is investigated using asymptotic methods and taking the boundary layer formed on the plate surface into account. It is shown that the effect of the boundary layer can be of two types depending on its profile. In the case of generalized convex profiles (characteristic of accelerated flow) supersonic and subsonic plate oscillations are stabilized and destabilized, respectively. In the case of profiles with a generalized inflection point located in the subsonic part of the layer (characteristic of homogeneous and decelerated flows) supersonic perturbations are destabilized in the thin boundary layer and stabilized when the layer is fairly thick; subsonic perturbations are damped.

Keywords: panel flutter, plate flutter, single-mode flutter, boundary layer, hydrodynamic instability.

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Earlier, the problem of stability of an elastic plate in potential inviscid gas flow was investigated using asymptotic methods [1]. It was shown that along with the well-known flutter of the coupled type, studied in detail [2], a single-mode flutter, which has not been studied earlier, can also occur. In this case the single-mode flutter cannot be revealed and investigated by means of the piston theory usually used in aeroelasticity and requires to use the exact potential gas flow theory or more complex theories. Later, these results were confirmed numerically [3] and experimentally [4].

In the previous studies the gas viscosity was neglected and flow was assumed to be homogeneous. However, in [5, 6] it was shown that flutter can be reduced or completely suppressed in the presence of a boundary layer. This result was obtained in investigating a particular boundary layer profile, namely, that with the one-seventh law. However, when aircrafts move, qualitatively different boundary layer profiles can develop in various parts of the surface depending on the flow conditions. In the present study the effect of the boundary layer of an arbitrary form on the single-mode flutter is investigated analytically.

1. FORMULATION OF THE PROBLEM

We will investigate the stability of an elastic, isotropically stretched plate exposed on one side to a plane-parallel gas flow. The plate has the shape of a strip and is embedded in an absolutely rigid plane. On its surface there is a boundary layer with dimensionless velocity and temperature profiles $u_0(z)$ and $T_0(z)$ nondimensionalized by the free-stream sonic speed and temperature, respectively.

The problem is solved in the plane formulation when there is no dependence on the y coordinate on the assumption that the undisturbed flow is plane-parallel, i.e., independent of x. We will assume that the flow is laminar and the Reynolds number $\mathrm{Re} \to \infty$. This is equivalent to the fact that the gas can be considered to be inviscid when perturbations are small and the effect of viscosity is expressed only in inhomogeneous velocity and temperature distributions.

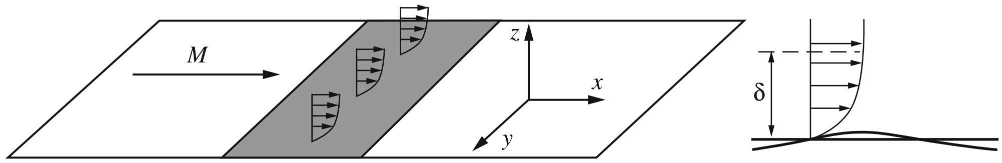

Fig. 1. Flow past a plate.

In dimensionless variables the equation of motion of the plate has the form:

$$D\frac{\partial^4 w}{\partial x^4} - M_w^2\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial t^2} + p = 0. \tag{1.1}$$

Here, w is the deflection of the plate, D is its stiffness, the parameter M_w characterizes its tension, and p is the gas pressure perturbation generated by the perturbation of the plate itself and, respectively, dependent on w. The problem also has two parameters L and μ which are equal, respectively, to the plate length divided by the thickness and to the ratio of the free-stream and plate-material densities. The detailed definition of the dimensionless parameters is given in [1].

The investigation consists of two stages. Initially, the plate is assumed to be infinite and its perturbations in the form of traveling waves $w(x,\, t) = e^{i(kx - \omega t)}$ are studied. Then, assuming that the plate is finite, we will construct its eigenfunctions in the form of a superposition of traveling waves which satisfies the boundary conditions. In this case we will use the global instability criterion [7].

2. DISPERSION RELATION FOR AN INFINITE PLATE IN GAS FLOW

In this section we will assume that the plate is infinite and the perturbation has the form $w(x,\, t) = e^{i(kx - \omega t)}$. In order to derive the dispersion relation between k and ω it is necessary to calculate the pressure perturbation acting on the plate. Propagation of perturbations in shear compressible gas flow can be described by the compressible Rayleigh equation [8]

$$\frac{d}{dz}\left(\frac{(u_0 - c)dv/dz - v du_0/dz}{T_0 - (u_0 - c)^2}\right) - \frac{1}{T_0}k^2(u_0\; c)v = 0. \tag{2.1}$$

Here, $c = \omega/k$ is the phase velocity of the wave and $v(z)$ is a perturbation of the vertical gas velocity component. The pressure perturbation can be expressed in terms of $v(z)$ as follows:

$$p(z) = \frac{\mu}{ik}\frac{(u_0 - c)dv/dz - v du_0/dz}{T_0 - (u_0 - c)^2}.$$

For the sake of brevity, we omitted the coefficient $e^{i(kx - \omega t)}$ of $v(z)$ and $p(z)$. The Rayleigh equation can have two singularities [8, 9]. The first singularity, located at a critical point z_c in which $u_0(z_c) = c$, leads to a singularity of the solution discussed below. The second singularity, located at a point in which $T_0(z) - (u_0(z) - c)^2 = 0$ and the phase velocity of the wave is equal to the local sonic speed, is removable.

We will consider the boundary conditions for the Rayleigh equation. We will impose the no-flow condition on the plate surface $z = 0$. The second condition will be imposed at the outer boundary layer edge $z = \delta$, where δ is the layer thickness divided by the plate thickness. We will assume that flow is homogeneous when $z > \delta$, $u_0 \equiv \mathrm{M}_\infty$ is the free-stream Mach number, and $T_0 \equiv 1$. The Rayleigh equation (2.1) can be reduced to an equation with constant coefficients and has the solution $v(z) = Ce^{\gamma z}$, where $\gamma = -\sqrt{k^2 - (\mathrm{M}_\infty k - \omega)^2}$. In this case we must choose the branch of the radical from the condition of perturbation damping as $z \to +\infty$ such that $\mathrm{Re}\,\gamma < 0$ when $\mathrm{Im}\,\omega \gg 1$. Outside the boundary layer this exponential solution must be matched with the solution inside the boundary layer. Thus, the boundary conditions have the form:

$$v = -i\omega \quad (z = 0), \qquad \frac{1}{v}\frac{dv}{dz} = \gamma \quad (z = \delta). \tag{2.2}$$

The solution of the Rayleigh equation can be constructed in the form of a convergent series in powers of k^2 [8]. Waves of length significantly larger than the boundary layer thickness are of practical interest for real plates. Therefore, we will assume that $|k| \ll 1$ and restrict out attention to the zeroth approximation. Neglecting the second term of the order of k^2 in (2.1), we have

$$\frac{d}{dz}\left(\frac{(u_0 - c)dv/dz - v du_0/dz}{T_0 - (u_0 - c)^2}\right) = 0$$

The general solution of this equation has the form:

$$v(z) = \left(c_1\left(\int_0^z \frac{T_0(\zeta)d\zeta}{(u_0(\zeta) - c)^2} - z\right) + c_2\right)(u_0(z) - c), \tag{2.3}$$

$$p(z) \equiv \frac{c_1\mu}{ik}. \tag{2.4}$$

We can readily see that (2.3) has a singularity at the critical point which cannot be removed within the framework of the inviscid approximation. When the viscosity tends to zero the solutions, which are the limit of the solutions of the viscous system, namely, an analog of the Orr–Sommerfeld equation in the incompressible case, can be constructed such that integration was carried out along a contour in the complex plane z bypassing the critical point from below [8, 9]. In particular, when $\operatorname{Im} c > 0$ (growing perturbations) integration can be carried out along the real axis z. In the case $\operatorname{Im} c \leq 0$ (neutral and damped perturbations) integration must be carried out in the complex plane bypassing the singularity.

The presence of the singularity leads to the fact that the function $v(z)$ is complex when $c \in \mathbf{R}$ and has a discontinuity at the critical point. If we take c_1 and c_2 so that $v \in \mathbf{R}$ when $z > z_c$, then v acquires suddenly (jumpwise) an imaginary part when $z < z_c$. Physically, this means that the perturbation phase changes jumpwise in passing through the critical point. In homogeneous flow there is no such a phase discontinuity but in our case this may lead to stabilization or destabilization of perturbations in the boundary layer.

Substituting (2.3) in the boundary conditions (2.2), from (2.4) we obtain the pressure perturbation on the plate surface

$$p(0) = -\mu\left(\left(\frac{\left(\mathrm{M}_\infty k - \omega\right)^2}{\sqrt{k^2 - \left(\mathrm{M}_\infty k - \omega\right)^2}}\right)^{-1} + \left(\int_0^\delta \frac{T_0(\zeta)\,d\zeta}{(u_0(\zeta) - c)^2} - \delta\right)\right)^{-1}.$$

For the further analysis it is convenient explicitly to distinguish the boundary layer thickness δ from the integral. For this purpose we make the change of variable $\zeta = \delta\eta$

$$\int_0^\delta \frac{T_0(\zeta)\,d\zeta}{\left(u_0(\zeta) - c\right)^2} = \delta\int_0^1 \frac{T_0(\eta)\,d\eta}{(u_0(\eta) - c)^2}.$$

In what follows, the functions $u_0(\eta)$ and $T_0(\eta)$ will be considered to be determining the boundary layer profile, η varying from 0 to 1.

Substituting the deflection $w = e^{i(kx - \omega t)}$ and the pressure perturbation $p = p(0)e^{i(kx - \omega t)}$ in (1.1), we obtain the dispersion relation

$$F(k,\ \omega) = \left(Dk^4 + M_w^2 k^2 - \omega^2\right) - \mu\left(\left(\frac{(\mathrm{M}_\infty k - \omega)^2}{\sqrt{k^2 - \left(\mathrm{M}_\infty k - \omega\right)^2}}\right)^{-1} + \delta\left(\int_0^1 \frac{T_0(\eta)\,d\eta}{\left(u_0(\eta) - c\right)^2} - 1\right)\right)^{-1} = 0. \quad (2.5)$$

When $\delta \to 0$ it is exactly transformed in the dispersion relation of a plate in homogeneous potential flow [1]

$$\left(Dk^4 + M_w^2 k^2 - \omega^2\right) - \mu\frac{\left(\mathrm{M}_\infty k - \omega\right)^2}{\sqrt{k^2 - (\mathrm{M}_\infty k - \omega)^2}} = 0.$$

The solution $\omega(k)$ of the dispersion relation can readily be obtained on the assumption that $\mu \ll |k|$, $\mu \ll |\omega|$. Using the Taylor expansion in μ, we have

$$\omega(k,\ \mu) = \omega(k,\ 0) + \mu\left.\frac{\partial\omega}{\partial\mu}\right|_{\mu=0} + o(\mu) = \omega(k,\ 0) - \mu\frac{\partial F}{\partial\mu}\Big/\left.\frac{\partial F}{\partial\omega}\right|_{\mu=0} + o(\mu).$$

Hence we have correct to small terms of the order of μ

$$\omega(k, \mu) = \omega(k, 0) - \frac{\mu}{2\omega(k, 0)}\left(\left(\frac{(\mathrm{M}_\infty k - \omega)^2}{\sqrt{k^2 - \left(\mathrm{M}_\infty k - \omega\right)^2}}\right)^{-1} + \delta\left(\int_0^1 \frac{T_0(\eta)\,d\eta}{\left(u_0(\eta) - c\right)^2} - 1\right)\right)^{-1}. \quad (2.6)$$

The expression in parentheses should be calculated for $\mu = 0$.

We note that the smallness of μ as compared with k and ω is a significant assumption on which too long waves must be eliminated from consideration. This assumption holds in studying the single-mode flutter but it is invalid for flutter of the coupled type since in this case $\mu \sim \omega$ [1].

3. EFFECT OF THE BOUNDARY LAYER ON THE GROWING WAVE

We will investigate the effect of the boundary layer on growth of the traveling waves in an infinite plate. We will distinguish three types of waves: 1) the wave is growing and the perturbation is supersonic with respect to flow $0 < c < \mathrm{M}_\infty - 1$ in the absence of the boundary layer; 2) the wave is neutral and the perturbation is subsonic $\mathrm{M}_\infty - 1 < c < \mathrm{M}_\infty + 1$ in the absence of the boundary layer; 3) the wave is damped and the perturbation is supersonic $c > \mathrm{M}_\infty + 1$ or $c < 0$ in the absence of the boundary layer.

Initially, we will consider the waves of the first type. We introduce the notation

$$A = \frac{\sqrt{k^2 - \left(\mathrm{M}_\infty k - \omega\right)^2}}{\left(\mathrm{M}_\infty k - \omega\right)^2} = \frac{\sqrt{1 - \left(\mathrm{M}_\infty - c\right)^2}}{k\left(\mathrm{M}_\infty - c\right)^2},$$

$$B = \delta\left(\int_0^1 \frac{T_0(\eta)\,d\eta}{\left(u_0(\eta) - c\right)^2} - 1\right),$$

then

$$\mathrm{Im}\,\omega(k,\ \mu) = -\frac{\mu}{2\omega(k,\ 0)}\mathrm{Im}(A + B)^{-1}.$$

Since the wave is growing when $\delta = 0$, then $\mathrm{Im}A = a > 0$ and $\mathrm{Re}A = 0$.

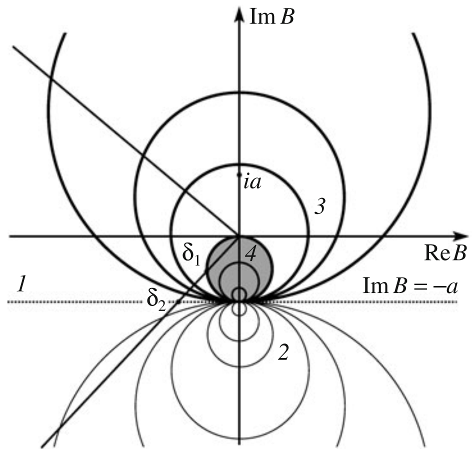

Fig. 2. Level lines of $\mathrm{Im}(A + B)^{-1}$: straight line *1* corresponds to $\mathrm{Im}(A + B)^{-1} = 0$; curves *2* correspond to the family of lines below straight line *1* where $\mathrm{Im}(A + B)^{-1} > 0$; curves *3* to the family of lines above straight line *1* when $\mathrm{Im}(A + B)^{-1} < 0$; and *4* corresponds to the region in which $\mathrm{Im}(A + B)^{-1} < \mathrm{Im}A^{-1} < 0$.

We can readily show that the set of B which are the level lines of $\mathrm{Im}(A + B)^{-1}$ are circles with the centers on the imaginary axis that pass through the point $(0; -a)$. If the second intersection of the circle and the imaginary axis lies above this point then $\mathrm{Im}(A + B)^{-1} < 0$, if below then $\mathrm{Im}(A + B)^{-1} > 0$. The level line $\mathrm{Im}(A + B)^{-1} = \mathrm{Im}A^{-1}$ is a circle that passes through the point $B = 0$. The case $\mathrm{Im}(A + B)^{-1} = 0$ (neutral perturbations) corresponds to a horizontal straight line that passes through the point $(0; -a)$ (Fig. 2).

We will fix the phase velocity c and consider the quantity $\mathrm{Im}(A + B)^{-1}$ as a function of the boundary layer thickness δ, assuming that the profiles $u_0(\eta)$ and $T_0(\eta)$ are given. The quantity A is independent of δ, while B depends linearly on δ. Then the values of B in the complex plane corresponding to various δ lie on a ray issuing from the origin. Two cases are possible. When $\mathrm{Im}B \geq 0$ this ray is directed upward or horizontally. In this case the wave traveling along the plate is growing for any boundary layer thickness, the wave growth rate being always lower than that in the absence of the boundary layer and tending monotonically to zero as $\delta \to \infty$.

In the second case $\mathrm{Im}B < 0$ and the ray is directed downward. When $0 < \delta < \delta_1$ the growth rate of the wave is positive in the presence of the boundary layer and higher then in its absence, namely, the ray intersects the circle $\mathrm{Im}(A + B)^{-1} = \mathrm{Im}A^{-1}$ when $\delta = \delta_1$. When $\delta_1 < \delta < \delta_2$ the growth rate is also positive but less than in the absence of the boundary layer, namely, the ray intersects the straight line $\mathrm{Im}(A + B)^{-1} = 0$ when $\delta = \delta_2$. Finally, the wave becomes damping when $\delta > \delta_2$.

The value of $\mathrm{Im}B$ which determines the qualitative behavior of the wave for various δ can be calculated in the explicit form. The integral in the expression for B has a singularity at $\eta = \eta_c$ which must be bypassed from below in the course of integration. We will expand the boundary layer profiles in the Taylor series in the neighborhood of $\eta = \eta_c$:

$$T_0(\xi) = T_{00} + T_{01}\xi + \ldots, \qquad u_0(\xi) = c + u_{01}\xi + u_{02}\xi^2/2 + \ldots, \qquad \xi = \eta - \eta_c.$$

Here, T_{0n} and u_{0n} are the nth derivatives of the functions at the critical point. We have

$$\frac{T_0(\xi)}{\big(u_0(\xi) - c\big)^2} = \frac{T_{00} + T_{01}\xi + \ldots}{\big(u_{01}\xi + u_{02}\xi^2/2 + \ldots\big)^2} = \frac{T_{00}}{u_{01}^2}\frac{1}{\xi^2} + \frac{1}{u_{01}^2}\left(T_{01} - T_{00}\frac{u_{02}}{u_{01}}\right)\frac{1}{\xi} + \text{reg. terms} \tag{3.1}$$

Since the term with $1/\xi$ is the source of the imaginary part of B, we obtain

$$\mathrm{Im}B = \frac{\pi\delta}{u_{01}^2}\left(T_{01} - T_{00}\frac{u_{02}}{u_{01}}\right) = -\pi\delta\frac{T_0^2}{u_0'^3}\left(\frac{u_0'}{T_0}\right)'.$$

Here, the prime denotes the derivative with respect to z at the critical point. Thus, in addition to the layer thickness, $\mathrm{Im}\,B$ is determined only by the local behavior of the velocity and temperature profiles in the neighborhood of the critical point. The values of $\mathrm{Re}\,B$ depend on all the regular terms of the expansion obtained above, i.e., they are determined by the profiles on the entire interval $[0;\ 1]$.

In the case of $\mathrm{Im}\,B < 0$ and $0 < \delta < \delta_1$ the growth rate of the wave is higher in the presence of the boundary layer then in its absence. The closer $|\mathrm{Re}\,B|$ to zero (the closer the direction of the ray to vertical), the higher the growth rate due to the boundary layer effect. In the limiting case $\mathrm{Re}\,B = 0$ and $\mathrm{Im}\,B \to -a$ the quantity $\mathrm{Im}(A + B)^{-1}$ together with $\mathrm{Im}\,\omega$ tend to infinity as $\delta \to \delta_1$.

As $c \to 0$ and $c \to M_\infty$, the behavior of $\mathrm{Re}\,B$ can be clarified in the generic case. After integrating (3.1), the principal term $\mathrm{Re}\,B$ is

$$\int\limits_{-\eta_c}^{1-\eta_c} \frac{T_{00}}{u_{01}^2}\frac{1}{\xi^2}\,d\xi = -\left.\frac{T_{00}}{u_{01}^2}\frac{1}{\xi}\right|_{-\eta_c}^{1-\eta_c} = -\frac{T_{00}(\eta_c)}{u_{01}^2(\eta_c)}\left(\frac{1}{1-\eta_c} + \frac{1}{\eta_c}\right).$$

Thus, as $c \to 0$ and $c \to M_\infty$ ($\eta_c \to 0$ and $\eta_c \to 1$, respectively), $\mathrm{Re}\,B \to -\infty$, i.e., the direction of the ray tends to horizontal.

4. EXAMPLE: PROFILES OF ACCELERATED AND DECELERATED FLOWS

We will consider the profiles developed in the self-similar boundary layers [10, Ch. XIII] characterizing by the parameter β. The flow is accelerated when $\beta > 0$, decelerated when $\beta < 0$, and homogeneous when $\beta = 0$. Analogs of such self-similar boundary layers develop in incompressible fluid when the free-stream velocity depends on the coordinate in accordance with the power law: $u_\infty = Cx^m$, where $\beta = 2m/(m+1)$. We will assume that the plate wavelength is much less than the characteristic dimension of the zone in which the free-stream velocity varies significantly. In other words, the decelerated or accelerated flow means the corresponding boundary layer profile considering, as before, the main flow to be homogeneous.

When the Prandtl number $\mathrm{Pr} = 1$ and the plate is heat-insulated the velocity distribution can be found by means of the solution $f(\xi)$ of the equation

$$f''' + ff'' = \beta(f'^2 - 1), \qquad f(0) = f'(0) = 0, \qquad f'(+\infty) = 1.$$

The velocity as a function of the self-similar variable ξ can be given by the expression $u_0(\xi) = f'(\xi)$. For each ξ the physical coordinate z can be restored as follows:

$$z = C\int\limits_0^\xi T_0(u_0(\xi))\,d\xi.$$

The temperature distribution $T_0(u_0)$ is given by the same expression as for adiabatic flow:

$$T_0(u_0) = 1 + \frac{\gamma - 1}{2}\left(M_\infty^2 - u_0^2\right). \tag{4.1}$$

We considered profiles *1–5* corresponding to $\beta = 2$, 0.5, 0, -0.14, and -0.199 (Fig. 3a). The last case corresponds to the limiting value of β when $du_0(0)/dz < 0$ for $\beta < -0.199$ and the boundary layer separates from the plate. The calculations were carried out for parameters corresponding to a steel plate in the atmosphere at the altitude of 3 km, $M_\infty = 1.6$, and

$$D = 23.9, \qquad M_w = 0, \qquad \mu = 0.00012, \qquad \gamma = 1.4. \tag{4.2}$$

As can be seen in Fig. 3, for profiles *1* and *2* $\mathrm{Im}\,B > 0$ and the growth in plate oscillations decays with increase in δ (Fig. 4). For profiles *3–5* $\mathrm{Im}\,B < 0$ in the supersonic zone, i.e., the boundary layer increases the oscillation amplitude when $\delta < \delta_1$ (Fig. 4). In this case $|\mathrm{Re}\,B|$ is fairly small on a significant interval $c < M_\infty - 1$ for profiles similar to profile *5* and $\mathrm{Im}\,\omega$ increases very substantially in the boundary layer.

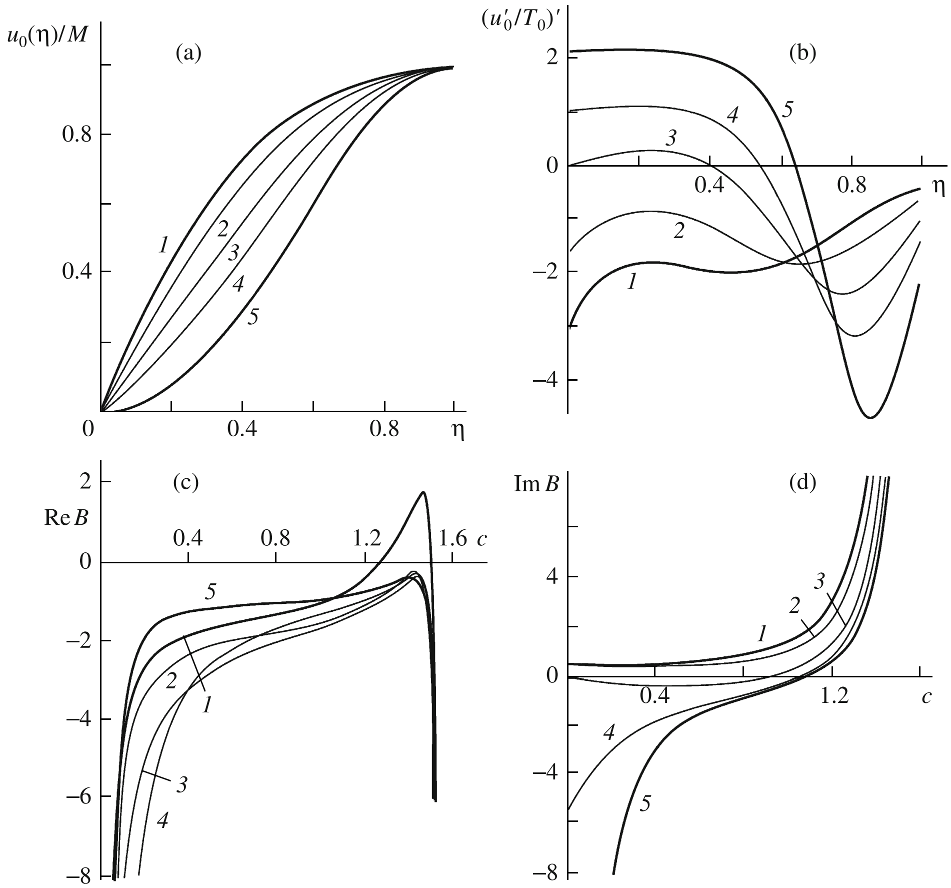

Fig. 3. Velocity profiles of decelerated and accelerated flows above a heat-insulated plate for parameters (4.2), $M_\infty = 1.6$, and $\Pr = 1$: velocity profiles (a); generalized curvature (b); ReB (c), and ImB (d); curves (*1–5*) correspond to $\beta = 2.0$, 0.5, 0.0, −0.14, and −0.199, respectively.

5. RELATION BETWEEN PLATE DESTABILIZATION BY THE BOUNDARY LAYER AND STABILITY OF THE BOUNDARY LAYER ITSELF

We will consider the effect of the boundary layer on the stability of a plate in connection with the stability of the absolutely rigid plate boundary layer itself. In this case the boundary conditions are as follows:

$$v = 0, \quad z = 0, \quad \frac{1}{v}\frac{dv}{dz} = \gamma, \qquad z = 1.$$

From (2.3) we can obtain the eigenvalue equation

$$\left(\frac{(M_\infty k - \omega)^2}{\sqrt{k^2 - (M_\infty k - \omega)^2}}\right)^{-1} + \delta\left(\int_0^1 \frac{T_0(\eta)\,d\eta}{\big(u_0(\eta) - c\big)^2} - 1\right) = 0$$

or

$$A + B = 0. \tag{5.1}$$

The solutions of this equation yield the spectrum of eigenvalues of c when $|k| \ll 1$; even if for a single of them $\operatorname{Im} c > 0$, the boundary layer is unstable.

If the boundary layer is unstable then there is its neutral perturbation lying on the boundary of the unstable interval of k. If this perturbation is supersonic, i.e., $c \in [0;\ M_\infty - 1]$, then $\operatorname{Re} A = 0$ and $\operatorname{Im} A > 0$. Then in Fig. 2 the ray is directed vertically downward. Thus, in order for longwave supersonic perturbations of the boundary layer to be unstable it is necessary that in Fig. 2 the ray to be directed vertically downward for a

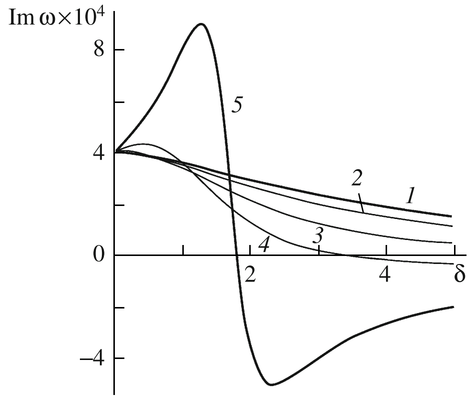

Fig. 4. Function $\mathrm{Im}\,\omega(\delta)$ for profiles (*1–5*) in Fig. 3 for parameters (4.2), $M_\infty = 1.6$, and $k = 0.06$.

certain $c \in [0;\ M_\infty - 1]$, whereas for significant enhancement of the plate oscillations it is sufficient only the proximity to the vertical.

The necessary condition of existence of a neutral subsonic perturbation ($M_\infty - 1 < c < M_\infty + 1$, $\mathrm{Im}A = 0$) follows from (5.1): $\mathrm{Im}B = 0$, i.e., there must exist a point in the subsonic part of the boundary layer at which $(u_0'/T_0)' = 0$. This point will be called the generalized inflection point of the profile. This condition is necessary and sufficient without the assumption on smallness of k [8, 9]. If a neutral subsonic perturbation exists (i.e., a generalized inflection point exists) then only the subsonic perturbations grow [11].

The profiles of Sect. 4 are stable with respect to longwave supersonic perturbations. In fact, as can be seen from Fig. 3, $\mathrm{Re}B$ does not vanish for any real values of $c < M_\infty - 1$, although it can be made fairly small when $\beta = -0.199$. This yields strong growth in $\mathrm{Im}\,\omega$ when $\delta < \delta_1$ (Fig. 4). However, the boundary layer is unstable with respect to subsonic perturbations when $\beta = 0$, -0.14, and -0.199 since in the subsonic part there is a point at which $\mathrm{Im}B = 0$ (Fig. 3d).

If the boundary layer profile is such that the generalized inflection point lies in the supersonic part of the boundary layer, such a layer can be stable but in this case it can yield an arbitrary strong increase in the plate oscillations. For this purpose we must have $\mathrm{Im}B < 0$ when $c \in [0;\ M_\infty - 1]$ and $\mathrm{Re}B$ must be negative and close to zero. In this case for the stability of the layer itself we must have $\mathrm{Re}B \neq 0$ when $c \in [0;\ M_\infty - 1]$ (supersonic perturbations) and $\mathrm{Im}B \neq 0$ when $c \in [M_\infty - 1;\ M_\infty]$ (subsonic perturbations).

As an example, we will consider the temperature profile (4.1) and the family of the velocity profiles parameterized by the parameter $s \in [0;\ 1]$ specified by the natural cubic spline passing through the following points:

$$\begin{gathered} u_0(0) = 0, \qquad u_0(0.105 - 0.045s) = 0.12M_\infty, \\ u_0(0.155 - 0.045s) = 0.21M_\infty, \qquad u_0(0.6) = 0.88M_\infty, \qquad u_0(1) = M_\infty. \end{gathered} \tag{5.2}$$

In Fig. 5a we have plotted the graphs of $u_0(\eta)$ for several s. When the profile is specified using a cubic spline, i.e., a function with continuous derivatives up to the second derivative, this implies that the spline can be approximated with an arbitrary accuracy (together with the first and second derivatives) by an analytic function.

In Figs. 5c and 5d we have plotted the graphs of $\mathrm{Re}B(c)$ and $\mathrm{Im}B(c)$ for these profiles when $c \in [0;\ M_\infty - 1]$. All the profiles are stable with respect to subsonic perturbations (Fig. 5b). When $s = 0$ the graph of the function $\mathrm{Re}B(c)$ is tangent to the horizontal axis, i.e., there is a neutral supersonic perturbation; the supersonic perturbations are damped when $s > 0$. Thus, all these profiles are stable.

However, as $s \to 0$, the ray in Fig. 2 tends to the vertical when $c \approx 0.21$. As $\delta \to \delta_1$, this leads to the unbounded rapid amplification of the plate oscillations. In Fig. 6 we have reproduced the results of calculations of $\mathrm{Im}\,\omega(\delta)$ for parameters (4.2), $M_\infty = 1.3$. We can see that the quantity $\mathrm{Im}\,\omega$ can increase by

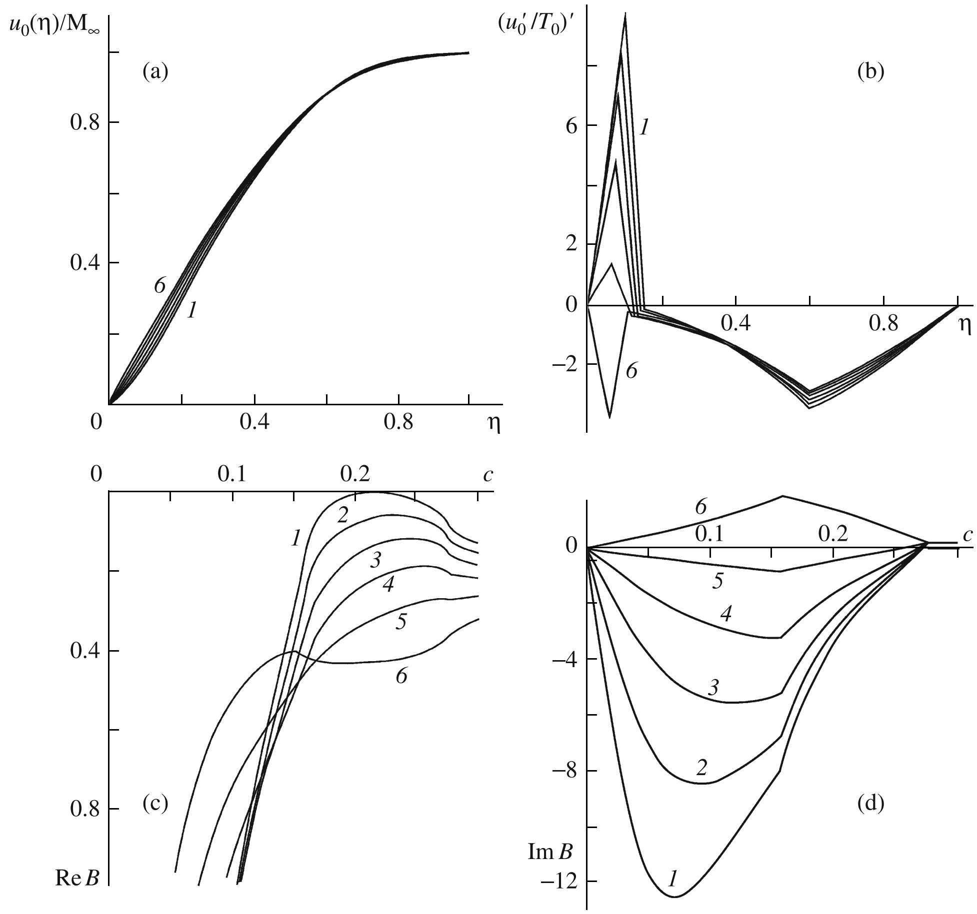

Fig. 5. Velocity profiles (5.2) for a heat-insulated plate and parameters (4.2), $M_\infty = 1.3$, and Pr = 1: velocity profiles (a); generalized curvature (b); ReB (c), and ImB (d); curves (*1–6*) correspond to $s = 0$, 0.2, 0.4, 0,6, 0.8, and 1, respectively.

a factor of more than two as compared with homogeneous flow when $s = 0.2$, by a factor of more than four when $s = 0.1$, and the value of $\operatorname{Im}\omega$ can be arbitrary large as $s \to 0$.

We note that the arbitrary rapid growth of the wave violates the assumption on which expansion (2.6) was obtained, namely, this expansion will not be linear in μ. Actually, we can only conclude that $\operatorname{Im}\omega \gtrsim \mu$. The situation is analogous to [12]: unbounded amplification exists also in flow without boundary layer as $c \to M_\infty - 1$, in this case the nonlinear expansion in μ yields $\operatorname{Im}\omega \sim \mu^{2/3}$.

6. EFFECT OF THE BOUNDARY LAYER ON THE NEUTRAL AND DAMPED WAVES

The question of the effect of the boundary layer when the wave in potential flow is growing, i.e. $0 < c < M_\infty - 1$, was considered above. We will now assume that the wave is damped, i.e., $M_\infty + 1 < c$ or $c < 0$. Then A is purely imaginary and $\operatorname{Im}A < 0$. There is no critical point and $\operatorname{Im}B = 0$. Then $0 < \operatorname{Im}(A + B)^{-1} < \operatorname{Im}A^{-1}$, i.e., the boundary layer always leads to certain reduction in the wave damping but cannot lead to its growth. The damped waves remain damped waves.

Let now a perturbation be neutral, i.e., $M_\infty - 1 < c < M_\infty + 1$ and $\operatorname{Im}A = 0$. There is no critical point when $M_\infty \le c \le M_\infty + 1$ and $\operatorname{Im}B = 0$. In the presence of the boundary layer the perturbation remains neutral.

When $M_\infty - 1 < c < M_\infty$ there is a critical point and $\operatorname{Im}B \neq 0$. For flows with the generalized-convex profile $(u_0'/T_0)' < 0$ and $\operatorname{sign}\operatorname{Im}B = -\operatorname{sign}((u_0'/T_0)') > 0$. Hence $\operatorname{Im}(A+B)^{-1} < 0$ and $\operatorname{Im}\omega > 0$, i.e., the wave is destabilized by the boundary layer. In this case the boundary layer itself is stable. This result is

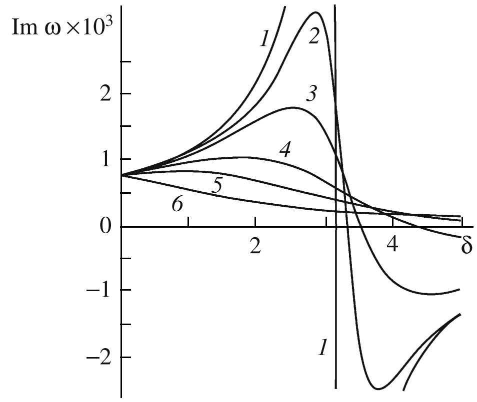

Fig. 6. Function $\mathrm{Im}\,\omega(\delta)$ for profiles (5.2) and parameters (4.2), $M_\infty = 1.3$, and $k = 0.043$ ($c \approx 0.21$): curves (*1–6*) correspond to $s = 0$, 0.1, 0.2, 0.4, 0.6, and 1, respectively.

similar to that obtained in [13] where it was shown that elasticity of the plate leads to destabilization of the system as $\mathrm{Re} \to \infty$ for the Blasius boundary layer profile in incompressible fluid.

Let now the profile have a generalized inflection point z_{infl} in the subsonic part of the layer. To be specific, we will assume that $(u_0'/T_0)' > 0$ when $z < z_{\mathrm{infl}}$ and $(u_0'/T_0)' < 0$ otherwise (such a situation is characteristic of the profiles on the flat heat-insulated plate). Then the waves which have the critical point when $z > z_{\mathrm{infl}}$, i.e., $M_\infty - 1 < u_0(z_{\mathrm{infl}}) < c < M_\infty$, are destabilized by the boundary layer since $\mathrm{sign}\,\mathrm{Im}\,B = -\,\mathrm{sign}((u_0'/T_0)') > 0$ and $\mathrm{Im}(A + B)^{-1} < 0$. To the contrary, the waves with the phase velocity $M_\infty - 1 < c < u_0(z_{\mathrm{infl}})$ are stabilized.

7. LARGE BUT FINITE PLATE

In this section we will investigate the effect of the boundary layer on the plate of large dimensions. Its eigenfunctions are constructed using the asymptotic method [7] in the form of superposition of waves in an infinite plate which satisfies the boundary conditions at the edges of a finite plate.

In accordance with [7], as the plate width $L \to \infty$, the limiting set of the eigenfrequency spectrum is independent of the boundary conditions at the plate edges and represents a curve Ω in the complex plate ω given by the equation $\mathrm{Im}\,k_2(\omega) = \mathrm{Im}\,k_3(\omega)$, where k_2 and k_3 are the branches of solutions of the dispersion relation of the unbounded system (2.5) corresponding to the forward and backward traveling waves (k_2 and k_3, respectively.) Since μ is a small parameter, then, assuming that $\omega = \omega_R + i\omega_I$, $\omega_I \ll 1$, we have the expansion

$$k_j(\omega_R + i\omega_I,\, \mu) = k_j(\omega_R,\, 0) + i\frac{\omega_I}{g_j} + \mu\Delta(k_j) + o(\omega_I,\, \mu), \qquad (7.1)$$

where

$$\Delta(k_j) = \frac{1}{2k_j\left(M_w^2 + 2Dk_j^2\right)}\left(\left(\frac{(M_\infty k_j - \omega)^2}{\sqrt{k_j^2 - \left(M_\infty k_j - \omega\right)^2}}\right)^{-1} + \delta\left(\int\limits_0^1 \frac{T_0(\eta)\,d\eta}{\left(u_0(\eta) - c_j\right)^2} - 1\right)\right)^{-1}.$$

Here, $c_j = \omega_R/k_j$ and $g_j = d\omega_R/dk_j$ are the phase and group velocities, respectively. Introducing the notation

$$g(\omega) = g_2 = -g_3 = \frac{2Dk_2^3 + k_2M_w^2}{\omega},$$

we obtain the equation of the asymptotic curve Ω

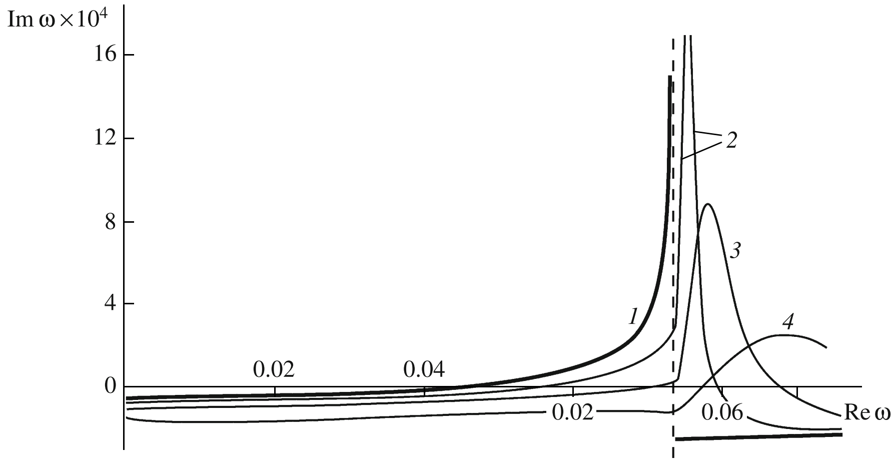

Fig. 7. Curve Ω for profile *1* in Fig. 3. Vertical broken straight line corresponds to transonic transition, $c = M_\infty - 1$: curves (*1–4*) correspond to $\delta = 0$, 0.5, 1, and 2, respectively.

$$\omega_I = -\frac{\mu g(\omega_R)}{2}\,\mathrm{Im}(\Delta(k_2(\omega_R)) - \Delta(k_3(\omega_R))) = -\frac{\mu}{2\omega_R}\,\mathrm{Im}\left((A + B)_2^{-1} + (A + B)_3^{-1}\right), \tag{7.2}$$

where A and B are the quantities determined earlier for the forward and backward traveling waves.

Thus, in gas flow with a boundary layer the plate oscillation growth rate is the arithmetic mean of the growth rates of the waves traveling upstream and downstream.

By analogy with the wave in the infinite plate, we will individually consider the oscillations depending on the type of the forward traveling wave, namely, growing, neutral, or damped (Sec. 3). Initially, we will consider the first case. If the boundary layer profile is generalized-convex, then the forward traveling wave remains growing but its growth rate decreases. A damped wave traveling upstream remains the damped wave but its damping rate also decreases. Generally speaking, the total action equal to the arithmetic mean of imaginary parts of the frequencies can lead to both increase and decrease of the oscillation growth rate of the finite plate. For typical boundary layer profiles the calculations show that the total action leads to a decrease in the oscillation growth rate or to damping (see Sec. 8).

When the boundary layer profile has a generalized inflection point and there is a zone in the supersonic zone with $\mathrm{Im}B < 0$, then the total action is growing over a certain frequency range where the forward traveling wave also grows in the boundary layer with $\delta < \delta_1$. This effect is particulary characteristic of the profiles of decelerated flow for which the growth of the forward traveling wave can be large (Sec. 4).

We will now consider the second class of the oscillations corresponding to subsonic waves traveling forward. If the profile is generalized-convex or has a generalized inflection point in the supersonic part of the layer then for the phase velocities close to $M_\infty - 1$ the growth rate of the forward traveling wave can be significant and greater than the damping of the backward traveling wave. In this case the total effect is growth of oscillations of the finite plate.

When the profile has a generalized inflection point in the subsonic part of the boundary layer located fairly far from the transonic point, then growth of the forward traveling wave is insignificant and not greater than damping of the backward traveling wave. The total effect is the damping of oscillations of the finite plate.

For the third class of oscillations both the forward and backward traveling waves are damped and such oscillations will be also damped in the finite plate.

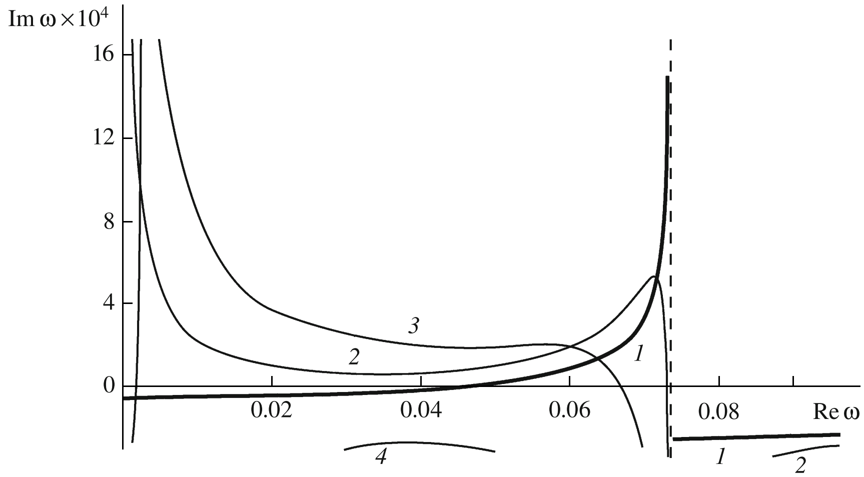

Fig. 8. Curve Ω for profile *5* in Fig. 3. Vertical broken straight line corresponds to transonic transition, $c = M_\infty - 1$: curves (*1–4*) correspond to $\delta = 0$, 0.5, 1, and 2, respectively.

8. EXAMPLES

We will summarize the results obtained in the previous section. In the case of the generalized-convex boundary layer profile the supersonic and subsonic oscillations of the plate are stabilized and destabilized, respectively. In the case of the profile with the generalized inflection point, in the subsonic part of the layer the supersonic oscillations grow more rapidly than in homogeneous flow when $\delta < \delta_1$ and are damped when $\delta > \delta_2$; the subsonic oscillations are always damped.

We will consider the generalized-convex profile of accelerated flow corresponding to $\beta = 2.0$ (profile *1* in Fig. 3). In Fig. 7 we have reproduced the results of calculations of Ω. We can see that for the supersonic oscillations $\mathrm{Im}\,\omega$ decreases and becomes negative with increase in the boundary layer thickness δ. At the same time, the subsonic perturbations are destabilized by the boundary layer.

In Fig. 8 we have reproduced the results of analogous calculations for $\beta = -0.199$ (profile *5* in Fig. 3). We can see that the instability of supersonic perturbations increases significantly when $\delta < \delta_1$: $\mathrm{Im}\,\omega$ increases for growing perturbations and the damped perturbations become growing. If $\delta > \delta_2$ (curve *2* in Fig. 8), the supersonic perturbations are completely stabilized. The subsonic perturbations remain damped since the generalized inflection point corresponds to $u_0 \approx 1.07M_\infty > M_\infty - 1$.

Summary. The effect of the boundary layer formed on the plate surface on the stability of waves in the infinite plate and on the single-mode flutter of a plate of large but finite dimensions is investigated.

In the case of generalized-convex profiles (characteristic of accelerated flow) the supersonic and subsonic plate oscillations are stabilized and destabilized, respectively.

In the case of profiles with the generalized inflection point lying in the subsonic part of the layer (characteristic of homogeneous and decelerated flows) the supersonic perturbations are destabilized when $\delta < \delta_1$ and stabilized when $\delta > \delta_2$; the subsonic perturbations are damped.

The relation between the effect of the boundary layer on the stability of the plate and the stability of the boundary layer itself is investigated. An example of stable boundary layers for which the growth rate of the plate oscillations can be made arbitrary large is given.

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