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Limit oscillatory cycles in the single mode flutter of a plate☆

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ABSTRACT

The development of the single mode flutter of an elastic plate in a supersonic gas flow is investigated in a non-linear formulation. In the case of a small depression in the instability zone, there is a unique limit cycle corresponding to a unique growing mode. Several new non-resonant limit cycles arise when a second increasing mode appears and the domains of their existence and stability are found. Limit cycles with an internal resonance, in which there is energy exchange between the modes, can exist for the same parameters. Relations between the amplitudes of the limit cycles and the parameters of the problem are obtained that enable one to estimate the risk of the onset of flutter.

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The problem of studying the stability of elastic plates in a supersonic gas flow arises in connection with the phenomenon of panel flutter, that is, the intensive vibrations of the individual skin panels of aircraft that are moving at high speed. This type of flutter is not immediately associated with wing and tail flutter but can also lead to fatigue damage accumulation and failure of the skin panels[1]

As a rule, piston theory, that is, the differential relation between the pressure perturbation and the flexure of the plate, is used [2–4] to calculate the perturbation of the pressure acting on it in studying the flutter of a plate. This theory is the limit of exact linearized potential flow theory when $M \to \infty$ or $\omega \to 0$ (M is the Mach number of the gas flow and ω is the vibrational frequency). It is assumed that piston theory works in practice starting approximately from $M \geq 1.7$ [3].

The principal disadvantage of piston theory lies in the fact that it can only describe the onset of one of the two forms of panel flutter, that is, coupled flutter. At low supersonic speeds $1 < M < 2$ The onset of another, single mode type of flutter occurs that can only be obtained using potential flow theory or more complex models. This type of flutter has been studied analytically (in the case of plates with large dimensions) [5] and numerically (in the case of arbitrary dimensions) [6] and experiments have been carried out [7] that confirm the existence of single mode flutter. The non-linear problem has been investigated and the amplitude of the limit cycle has been obtained in the case when only one mode is growing[8] It has been shown that the increase in the amplitude when there is a depression in the instability zone occurs far more rapidly than in the case of coupled flutter. However, the domain of single-mode flutter is small and, when a small increase in the flow velocity occurs, several modes at once become growing modes. The limit cycles that occur in the case of flutter in several modes simultaneously are studied below.

1. Formulation of the problem.

The non-linear vibrations of a tensioned elastic plate, over one side of which there is a plane-parallel supersonic flow of perfect inviscid gas (Fig. 1), are studied in the case of single-mode flutter. *A* constant pressure, equal to the unperturbed pressure of the flow, is applied to the other side of the plate. The non-linearity of the problem is caused by the geometric non-linearity of the plate, that is, by the membrane stresses arising under bending (Karman's model of large deflections). The perturbation in gas pressure acting on the plate is assumed to depend linearly on the deflection, since the aerodynamic non-linearity only has a considerable effect on the vibrations of the plate at very high (of the order of 10) Mach numbers M and in the transonic region.[9]

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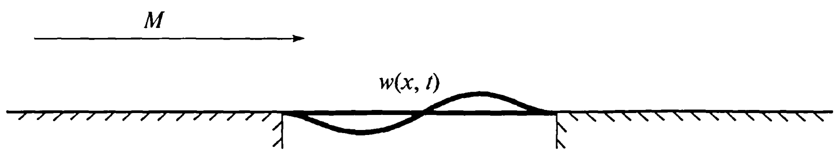

Fig. 1.

The problem is considered in a two-dimensional formulation. After expressing the pressure perturbation in terms of the deflection of the plate, the problem becomes a one-dimensional problem with one unknown function, the deflection. In dimensionless variables, the equation of motion of the plate has the form (Ref. 10, §24)

$$D\frac{\partial^4 w}{\partial x^4}-\left(M_w^2+K\frac{1}{2L}\int\limits_{-L/2}^{L/2}\left(\frac{\partial w(\xi)}{\partial \xi}\right)^2 d\xi\right)\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial t^2}+P\{w\}=0$$

$$D=\frac{E}{12(1-\nu^2)a^2\rho_m},\quad M_w=\frac{\sqrt{\sigma/\rho_m}}{a},\quad K=\frac{E}{(1-\nu^2)a^2\rho_m}=12D,\quad L=\frac{L_w}{h},\quad M=\frac{u}{a},\quad \mu=\frac{\rho}{\rho_m} \tag{1.1}$$

Here, $w(x, t)$ is the deflection of the plate normalized by its thickness, E, ν and ρ_m are Young's modulus, Poisson's ratio and the density of its material, L_w and h are the width and thickness of the plate, σ is the tensile stress, u and ρ are the velocity and density of the gas,. and a is the gas sound velocity. The plate tension (the coefficient of $\partial^2/\partial x^2$) consists of two parts: the first term is the constant tension applied to the plate and the second term is the non-linear tension arising when the plate is deflected. The structural damping of the plate is neglected (it can be taken into account by reducing the aerodynamic growth rate introduced below by the magnitude of the damping factor). The parameters D and L are the dimensionless stifness and width of the plate, M_w and K characterize its tension and non-linearity, and M and μ are the Mach number and the dimensionless gas density. The expression for K is written assuming that the edges of the plate are not displaced when the plate. bends otherwise, $K < 12D$. The operator $P\left\{w(x, t)\right\}$ is the perturbation in the gas pressure. The plate occupies the domain $-L/2 \le x \le L/2$ and the clamping or pinning conditions on the edges are specified.

2. Amplitude equations

The problem is solved using the Bubnov-Galerikin method. We expand the solution (1.1) in the orthonormal eigenfunctions of the plate in a vacuum:

$$w(x,t)=\sum_{j=1}^{\infty} W_j(x)A_j(t) \tag{2.1}$$

corrected with unknown amplitudes $A_j(t)$. After applying the Bubnov-Galerkin procedure, we obtain the system of equations in $A_n (n = 1, 2, ...)$

$$\frac{d^2A_n}{dt^2}+\omega_{0n}^2A_n+K\sum_{m,k,j=1}^{\infty} a_{mk}a_{jn}A_mA_kA_j+\sum_{j=1}^{\infty}P_{jn}=0 \tag{2.2}$$

where

$$a_{jn}=\frac{1}{\sqrt{2L}}\int\limits_{-L/2}^{L/2}\frac{dW_j}{dx}\frac{dW_n}{dx}dx,\quad P_{jn}(t)=\int\limits_{-L/2}^{L/2}P\{W_jA_j\}W_n dx$$

Here, $a_{jn} = a_{jj} > 0$ and, in the special case of hinged support along both edges, $a_{jn} = 0$ when $j \neq n$.

It has been shown [8] that, in the case of single-mode flutter, when the effect of the flow on the natural modes of the plate is small:

$$\frac{|P_{jn}|}{|P_{nn}|}\ll 1,\ j\neq n,\quad P_{nn}(t)\approx -2p_{n2}(\omega)\frac{dA_n(t)}{dt} \tag{2.3}$$

The meaning of the last relation lies in the fact that, in the case of single-mode flutter, the action of the flow reduces to an aerodynamic enhancement (or damping) of the vibrations. The aerodynamic enhancement coefficient $p_{n2}(\omega)$ depends on the characteristic frequency of the vibrations and there is a frequency range $\omega_n'' < \omega < \omega_n'$, where its sign is positive (the qualitative form of the function $p_{n2}(\omega)$ is shown in Fig. 2) that corresponds to single-mode instability in the linear approximation. The values of ω_n' and ω_n'' are different for the different modes. The explicit formula

$$\omega_n' = (M-1)\chi_n/L \tag{2.4}$$

is subsequently required where χ_n are constants occurring in the formula for the n-th natural frequency of the plate

$$\omega_{0n}^2 = D(\chi_n/L)^4 + M_w^2(\chi_n/L)^2$$

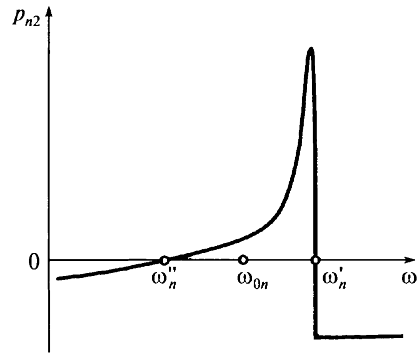

Fig. 2.

that depend on the boundary conditions.

Grouping terms with different powers of A_n in Eqs (2.2) and taking account of the approximate equalities (2.3), we obtain

$$\frac{d^2A_n}{dt^2} - 2p_{n2}(\omega)\frac{dA_n}{dt} + \left(\omega_{0n}^2 + K\sum_{\substack{m,k=1\\ m,k\neq n}}^{\infty}(a_{mk}a_{nn} + 2a_{mn}a_{kn})A_mA_k\right)A_n$$
$$+ 3K\left(\sum_{\substack{m=1\\ m\neq n}}^{\infty} a_{mn}a_{nn}A_m\right)A_n^2 + Ka_{nn}^2A_n^3 + K\sum_{\substack{m,k,j=1\\ m,k,j\neq n}}^{\infty} a_{mk}a_{jn}A_mA_kA_j = 0 \tag{2.5}$$

3. One linearly growing mode

Assuming that $|A_n| << |A_1|$, $n > 1$, all the terms containing amplitudes with an index greater than 1 in Eq. (2.5) for A_1 can be discarded. It is then written in the form

$$\frac{d^2A_1}{dt^2} - 2p_{12}(\omega)\frac{dA_1}{dt} + \omega_{01}^2A_1 + Ka_{11}^2A_1^3 = 0 \tag{3.1}$$

The limit cycles of this equation in the second approximation:

$$A_1(t) = C_1\cos\omega t + C_3\cos 3\omega t; \quad C_1 = \sqrt{\frac{4(\omega^2-\omega_{01}^2)}{3Ka_{11}^2}}, \quad C_3 = \frac{Ka_{11}^2}{4(9\omega^2-\omega_{01}^2)}C_1^3$$
$$p_{12}(\omega) = 0 \Rightarrow \omega = \omega_1', \quad \omega = \omega_1'' \tag{3.2}$$

have been found [8] by the harmonic balance method.

Since, $\omega_1'' < \omega_{01} < \omega_1''$, in the case when the first mode is an increasing mode in the linear approximation, we obtain from relations (3.2) that vibrations with a frequency ω_1'' are impossible and there is only one limit cycle that has a frequency ω_1'.

We will now consider the process of the enhancement of the vibrations and imagine that a small initial perturbation of the first mode that increases with time $p_{12}(\omega_{01}) > 0$ is excited in the plate. On account of the non-linear term in Eq. (3.1), the amplitude and frequency are related to one another, and the frequency, following the amplitude, also starts to increase. An increase in the frequency implies movement to the right along the curve $p_{12}(\omega)$ (Fig. 2). Since, when no account is taken of the action of the gas, the solutions of Eq. (3.1) have the form of neutral vibrations with arbitrary amplitude, non-linearity in itself cannot lead to a cessation in the growth of the amplitude. As a result, as long as $p_{12}(\omega) > 0$ the amplitude and, consequently, also the frequency will continue to increase. When the frequency ω reaches the value ω_1', the value of $p_{12}(\omega)$ becomes equal to zero and the enhancement gives way to a neutral vibration. There will be no principal internal resonance, since $\omega_1' < \omega_n' < \omega_{0n}$, $n > 1$. At the same time, fractional resonances are not excluded and they are considered below.

The arguments presented are clearly illustrated by the energy equation. We multiply Eq. (3.1) by dA_1/dt and transform it to the form

$$\frac{d}{dt}\left(\frac{1}{2}\left(\frac{dA_1}{dt}\right)^2 + \frac{1}{2}\omega_{01}^2A_1^2 + Ka_{11}^2\frac{A_1^4}{4}\right) = 2p_{12}(\omega)\left(\frac{dA_1}{dt}\right)^2 \tag{3.3}$$

The left-hand side of the equality obtained is the change in the total vibrational energy. When $p_{12}(\omega) > 0$ it increases, when $p_{12}(\omega) < 0$ is decreases and it is only in the case when $p_{12}(\omega) = 0$ that the vibrations are neutral.

It follows at once from Eq. (3.3) that the limit cycle with a frequency ω_1' obtained is stable, since an increase (decrease) in the amplitude leads to an increase (decrease) in the frequency and to the opposite effect on the part of the pressure, that is, to a decrease (increase) in the energy and amplitude.

Note that the relation between the amplitudes (3.2) and frequency is the same as that in the case of the non-linear vibrations of a plate in vacuo. The difference lies in the fact that the frequency of the vibrations in vacuo can be arbitrary but in the flow it is determined by the condition $\omega = \omega_1'$.

4. Non-resonance multifrequency vibrations

Specific calculations show that the range of Mach numbers over which only one mode is growing is extremely narrow. We will consider the case when a second mode also becomes growing. Keeping the two amplitudes A_1 and A_2 in equalities (2.5), we obtain the system of equations

$$\mathscr{L}_1 A_1 + K(a_{11}a_{22} + 2a_{12}^2)A_2^2 A_1 + 3Ka_{11}a_{21}A_2A_1^2 + Ka_{11}^2A_1^3 + Ka_{22}a_{12}A_2^3 = 0$$
$$\mathscr{L}_2 A_2 + K(a_{11}a_{22} + 2a_{12}^2)A_1^2 A_2 + 3Ka_{22}a_{12}A_1A_2^2 + Ka_{22}^2A_2^3 + Ka_{11}a_{12}A_1^3 = 0$$
$$\mathscr{L}_j = \frac{d^2}{dt^2} - 2p_{j2}(\omega)\frac{d}{dt} + \omega_{0j}^2, \quad j = 1,2 \tag{4.1}$$

As earlier[8] we assume that each of the steady- vibration modes is close to a harmonic vibration. The frequencies of the fundamental harmonics of the first and second modes are denoted by ω_1 and ω_2. Since system (4.1) does not separate, the solution for each amplitude contains not only the subharmonics of the fundamental frequency but also all possible harmonics with frequencies of the form $m\omega_1 \pm n\omega_2$, $m, n \in \mathbb{N}$. Then, omitting the exact form of the solution on account of its length, we write out the harmonics that are important in the second approximation:

$$A_j(t) = C_{j1}\cos(\omega_j t + (j-1)\varphi) + C_{j3}\cos(3\omega_j t + 3(j-1)\varphi)$$
$$+ Q_{j1}\cos((2\omega_2 - \omega_1)t + 2\varphi) + Q_{j2}\cos((2\omega_2 + \omega_1)t + 2\varphi)$$
$$+ Q_{j3}\cos((2\omega_1 - \omega_2)t - \varphi) + Q_{j4}\cos((2\omega_1 + \omega_2)t + \varphi)$$
$$+ Q_{j5}\cos(\omega_{3-j}t + (2-j)\varphi) + Q_{j6}\cos(3\omega_{3-j}t + 3(2-j)\varphi) + \dots, \quad j = 1,2$$

Here, v is the arbitrary phase shift between the first and second modes and the series in the remaining harmonics are replaced by the dots. Assuming that the amplitudes of the fundamental harmonics C_{11} and C_{21} are much greater than those of the remaining harmonics, we substitute these expressions into system (4.1) and equate the coefficients of $\cos\omega_1 t$, $\sin\omega_1 t$, $\cos(\omega_2 t+\varphi)$, $\sin(\omega_2 t+\varphi)$. We then obtain the system for the determining the amplitudes and frequencies of the fundamental harmonics.

We note the properties of the solutions of this system. The equations for the amplitudes (obtained from the equalities of the cosine coefficients) are the same as for the vibrations of the plate in vacuo. Taking account of the action of the gas only reduces to the choosing the specific frequencies $\omega_1 = \omega_1'$ and $\omega_2 = \omega_2'$ that are solutions of the equations $p_{12}(\omega_1) = 0$ and $p_{22}(\omega_2) = 0$ such that $\omega_1' > \omega_{01}$ and $\omega_2' > \omega_{02}$. The amplitudes of the higher harmonics are explicitly expressed in terms of C_{11} and C_{21} but the corrections introduced by them are small.

The system of equations for C_{11} and C_{21} has three non-trivial solutions. The two solutions

$$C_{(3-j)1} = 0, \quad C_{j1}^2 = \frac{4(\omega_j^2 - \omega_{0j}^2)}{3\kappa_j}, \quad \omega_j = \omega_j'; \quad \kappa_j = Ka_{jj}^2$$

describe the single-frequency first (j =1) and second (j = 2) vibration modes. The tr third solution is the two-frequency vibration:

$$C_{j1}^2 + \frac{2\eta_j}{3}C_{(3-j)1}^2 = \frac{4(\omega_j^2 - \omega_{0j}^2)}{3\kappa_j}, \quad \omega_j = \omega_j', \quad \eta_j = \frac{a_{11}a_{22} + 2a_{12}^2}{a_{jj}^2}, \quad j = 1,2 \tag{4.2}$$

Solving system (4.2) for the amplitudes, we obtain

$$C_{j1}^2 = \frac{12(\omega_j^2 - \omega_{0j}^2) - 8\eta_{3-j}(\omega_{3-j}^2 - \omega_{0(3-j)}^2)}{9\kappa_j - 4\kappa_{3-j}\eta_{3-j}^2} > 0, \quad \omega_j = \omega_j', \quad j = 1,2 \tag{4.3}$$

These inequalities are a condition for the existence of a two-frequency limit cycle; single frequency limit cycles obviously always exist. We will now investigate these conditions assuming, for simplicity, that identical boundary conditions are specified on the leading and trailing edges of the plate. Then, $a_{12} = 0$.

Substituting expressions (2.4) into inequalities (4.3), we rewrite them in the form

$$N_1 - \frac{4}{9}\zeta N_2 > 0, \quad N_1 + \zeta N_2 > 0, \quad N_j = (M-1)^2 - (M_j^* - 1)^2, \quad \zeta = \frac{3}{2}\frac{a_{11}}{a_{22}}\frac{\chi_2^2}{\chi_1^2} > 1 \tag{4.4}$$

Here, $M = M_j^*$ are the boundaries of stability of the first mode (j = 1) and second mode (j = 2). In the case of the pinning conditions for the plate edges $a_{11}/a_{22} = 0.25$, $x_2^2/x_1^2 = 4$, $\zeta = 3/2$ and, for the clamping condition, $a_{11}/a_{22} = 0.267$, $x_2^2/x_1^2 = 2.761$, $\zeta = 1.106$. It is easily seen that, under these conditions, inequality (4.4) is always satisfied since $M_1^* < M_2^*$.

The second inequality of (4.4) is not satisfied when $M < M_2^*$, that is, a two-frequency limit cycle does not always exist. Solving this inequality for M, we obtain

$$M \geq 1 + \sqrt{\frac{\zeta(M_2^* - 1)^2 - (M_1^* - 1)^2}{\zeta - 1}} \quad (4.5)$$

As an example, consider a metal plate in the air flow at sea level:

$$D = 23.9, \quad L = 300 \quad (4.6)$$

Then, $M_1^* \approx 1.051$, $M_2^* \approx 1.102$ in the case of pinning and $M_1^* \approx 1.077$, $M_2^* \approx 1.128$ in the case of clamping. The condition for the existence of the two-frequency limit cycle (4.5) has the form $M \geq 1.162$ and $M \geq 1.34$ respectively.

We will now consider the stability of the limit cycles and write down the energy equation for the plate. To do this, we multiply the first equation of (4.1) by dA_1/dt, the second equation by dA_2/dt and add them together:

$$\frac{d}{dt}\left(\frac{1}{2}\left(\frac{dA_1}{dt}\right)^2 + \frac{1}{2}\left(\frac{dA_2}{dt}\right)^2 + \frac{1}{2}\omega_{01}^2 A_1^2 + \frac{1}{2}\omega_{02}^2 A_2^2 + \frac{Ka_{11}^2}{4}A_1^4 + \frac{Ka_{22}^2}{4}A_2^4 \right.$$
$$\left. + \frac{1}{2}K(a_{11}a_{22} + 2a_{12}^2)A_1^2A_2^2 + Ka_{12}a_{11}A_1^3A_2 + Ka_{12}a_{22}A_1A_2^3\right)$$
$$= 2p_{12}(\omega_1)\left(\frac{dA_1}{dt}\right)^2 + 2p_{22}(\omega_2)\left(\frac{dA_2}{dt}\right)^2 \quad (4.7)$$

The expression on the left-hand side is the derivative of the total energy of the plate (the sum of the kinetic energy and potential energy) and the right-hand side is the work performed by the gas flow on the plate.

Suppose only single-frequency limit cycles exist. In the same way as in Section 3 it is proved that they are both stable with respect to perturbations of the mode in which the vibrations occur: an increase in its amplitude leads to a reduction in the total energy and, as a consequence in its amplitude, while a decrease in the amplitude leads to an increase in the energy and amplitude.

We will now consider the perturbations of an initially quiescent mode. We will first consider the first mode and perturb second mode by imparting a small amplitude $C_{21} \approx 0$ to it. Its frequency ω_2 is determined from the first equation of (4.2) for $j = 2$. Since the inequality (4.3) when $\omega_2 = \omega_2'$ is not satisfied for $j = 2$ (otherwise a two-frequency limit cycle would exist), then $\omega_2 > \omega_2'$ whence $p_{22}(\omega_2) < 0$. From (4.7), we then obtain that the vibrations conforming to the second mode decay and the limit cycle of the vibrations conforming to the first mode is stable. We now consider the vibrations conforming to the second mode and perturb the first mode, imparting a small amplitude $C_{11} \approx 0$ to it. Its frequency ω_1 is determined from the first equation of (4.2) for $j = 1$. Since, when $\omega_1 = \omega_1'$, inequality (4.3) is strictly satisfied for $j = 1$, then $\omega_1 = \omega_1'$. However, this means that $p_{12}(\omega > 0)$ and the vibrations conforming to the first mode are growing. The limit cycle of the vibrations conforming to the second mode is therefore unstable.

It can similarly be proved that, if a two-frequency limit cycle exists, it is stable, and both single-frequency limit cycles are unstable with respect to perturbations of the initially quiescent mode. Hence, only one non-resonance limit cycle is stable and the number of modes participating in it is a maximum among the cycles that exist for the given problem parameters.

When M is increased and some mode emerges from the zone of instability $\omega_{0n} < \omega_n''$, an unstable limit cycle appears with a frequency of the vibrations conforming to this mode ω_n'', that separates the domains of attraction of the two stable cycles in one of which this mode participates and, in the other of which, it does not participate. The appearance of this cycle has been considered [8] in the case of one growing mode.

In the case of arbitrary boundary conditions and instability with respect to three or more modes, there are also limit cycles consisting of a different number of modes, but their stability intervals can interact in a more complex manner than in the case of two modes. Note that the order of the amplitudes in nonresonance multi-frequency limit cycles is the same as in single-frequency limit, cycles, since they are all described by equations of the type (4.2)

5. Multifrequency vibrations with an even internal resonance

It is well known that, in the case of an external periodic force acting on a linear system, a resonance vibration only occurs if its frequency coincides with a natural frequency. In addition to the fundamental resonance, the existence of non-linearity also generates fractional resonance vibrations with frequencies of the form $p\omega/q$, where p and q are integers.[11–13] If a system consists of several coupled subsystems (like a plate, each mode of which can be considered as a separate subsystem), they can resonate with one another and generate vibrations with an internal resonance.

We shall now consider vibrations for which there is a 1:2 internal resonance and, to be specific, we shall assume that the first and second modes resonate. For simplicity, we shall confine ourselves in this Section to the case when identical boundary conditions are specified on the edges of the plate, so that $a_{12} = 0$. We shall also assume that the boundary conditions differ from the pinning conditions as, otherwise, $\omega_2'(M) = 2\omega_1'(M)$ and the difference between the resonance and non-resonance limit cycle considired above vanishes.

We first note an important property of an even resonance. The cubic non-linearity in the case of such vibrations $(a\cos\omega_1 t + b\cos\omega_2 t)^3$ gives, among other things, a harmonic with a frequency $2\omega_1 - \omega_2$. In the case of 1:2 resonance, when $\omega_2 = 2\omega_1$, this harmonic is a constant deflection of the plate from the equilibrium position. As will be seen below, it ensures that there is energy exchange between the modes in the case of even resonance and it must be taken into account. We will therefore seek a solution in the form

$$A_1 = C_{10} + C_{11}\cos\omega t + C_{12}\cos(2\omega t + \varphi) + \ldots$$
$$A_2 = C_{20} + C_{21}\cos(\omega t + \alpha) + C_{22}\cos(2\omega t + \beta) + \ldots \quad (5.1)$$

We will assume that, under the action of the flow, the plate executes quasiharmonic vibrations conforeming to the first mode, $|c_{10}|, |c_{12}| << |c_{11}|$, which excite resonance vibrations conforming to the second mode with a doubled frequency, $|c_{20}|\,|c_{12}| << |c_{22}|$.

Substituting this solution into Eq. (4.1) and equating the coefficients for all possible sines and cosines, we obtain a system of equations for determining the amplitudes, frequencies and phase shifts. we need equations for the amplitudes of the fundamental harmonics in the first approximations which follow from the cosine relations, the solution of which has the form

$$C_{11}^2 = \frac{12}{5\kappa_1}\left(\left(1-\frac{8}{3}\eta_2\right)\omega^2 - \left(\omega_{01}^2 - \frac{2}{3}\eta_2\omega_{02}^2\right)\right)$$

$$C_{22}^2 = \frac{12}{5\kappa_2}\left(\left(4-\frac{2}{3}\eta_1\right)\omega^2 - \left(\omega_{02}^2 - \frac{2}{3}\eta_1\omega_{01}^2\right)\right) \tag{5.2}$$

and the expression for the amplitude of the deflection C_{20} in the second approximation

$$C_{20} = -\frac{\eta_2 C_{11}^2 C_{22}\cos\beta}{6C_{22}^2 + 2\eta_2 C_{11}^2 + 4\omega_{02}^2/\kappa_2} \tag{5.3}$$

In the second approximation, the equations following from the equalities of the coefficients of the sines of the fundamental harmonics appear as:

$$2p_{12}(\omega)\omega = \kappa_1\eta_1 C_{20}C_{22}\sin\beta \tag{5.4}$$

$$4p_{22}(2\omega)\omega C_{22} = -\frac{1}{2}\kappa_1\eta_1 C_{11}^2 C_{20}\sin\beta \tag{5.5}$$

In particular, the energy balance equation

$$p_{12}(\omega)C_{11}^2 + 4p_{22}(2\omega)C_{22}^2 = 0 \tag{5.6}$$

follows from them and the physical meaning of this equation is as follows: the amount of energy transferred from the flow to the plate via the first mode after one period of the vibrations is equal to the amount of energy transferred from the plate back into the flow via the second mode. According to equality (4.7), the overall work of the flow on the plate after one period of the vibrations is equal to zero.

The system of equations for determining the amplitudes, frequencies and phase β consists of Eqs (5.2), (5.3), (5.4) and (5.6). The equations following from the equalities of the coefficients of the cosines and sines of the subharmonics and the constant deflection C_{10} are homogeneous and have the solution $C_{10} = C_{12} = C_{21} = 0$.

We will now prove that, for sufficiently small μ, a stable limit cycle arises at the onset of a resonance of the second mode but prior to the onset of non-resonance multi-frequency vibrations. Prior to the onset of the resonance C_{22} = 0, the amplitude C_{11} is determined by expression (3.2) and the plate executes single-frequency vibrations, $\omega = \omega_1'$. The resonance solution arises at a frequency $\omega_1' > \hat{\omega}$, when the right-hand side of the second equality of (5.2) becomes positive. Here,

$$\hat{\omega} = \sqrt{\left(\omega_{02}^2 - \frac{2}{3}\eta_1\omega_{01}^2\right)\left(4-\frac{2}{3}\eta_1\right)^{-1}}$$

In this case, a limit cycle with an intermediate frequency exists. Actually, when $\omega = \hat{\omega}$, the left-hand side of Eq. (5.6) is positive and, when $\omega = \omega_1'$, it is negative. Condition (5.6) is therefore satisfied for some frequency. $\omega \in (\hat{\omega}, \omega_1')$.

The mutual arrangement of the curves C_{11}, C_{22}, $p_{12}(\omega)$ and p_{22} (2ω) on the section $[\hat{\omega}_1\omega_1']$ is shown in Fig. 3 and the graphical solution of Eq. (5.6) for different values of ω_1' is shown in Fig. 4.

Then, a solution β of Eq. (5.4) exists for sufficiently small μ. Actually, the right-hand side is independent of μ and the left-hand side can always be made as small as desired by taking a sufficiently small μ. As a result, in the case of small μ, Eq. (5.4) is solvable for β and a limit cycle with an internal 1:2 resonance exists.

It is similarly proved that a resonance limit cycle arises for a fixed μ if the amplitude C_{22} becomes sufficiently large. We note that Eq.(5.4) does not have solutions β for a fixed μ and a sufficiently small amplitude C_{22}. This means that a resonance limit cycle does not arise when $M = \hat{M}$, where $\omega_1'(\hat{M}) = \hat{\omega}$, but it arises at a higher value of M, and at once has a finite amplitude C_{22}.

We will now prove that the limit cycle constructed for small μ is stable. We first consider the deviation of one of the frequencies from a ratio of 1:2. With respect to the equations describing the vibrations conforming to the second mode when a_{12} = 0, the excitation of this resonance is a parametric resonance caused by the presence of the second term in the second of equations (4.1). It is well known (Ref. 12, §17) that a parametric resonance is stable in the case of a fixed non-linearity coefficient and damping that tends to zero. In other words, for small μ the deviation of one of the frequencies from a ratio of 1:2 is automatically "corrected" with the passage of time, that is, there is frequency synchronization. It can therefore be assumed that this ratio is fixed and the amplitudes are, correspondingly, rigidly connected with the frequency by Eq. (5.2).

We will now consider a small change in the frequency ω without changing the 1:2 ratio. By virtue of equality (4.7), the left-hand side of Eq. (5.6) is the change in the total energy of the plate after one period of the vibrations. We now increase the frequency of the limit cycle ω (which leads to an increase in the amplitudes of the two harmonics). Then, by construction, the left-hand side of (5.6) becomes negative, that is, there is a flow of vibrational energy from the plate into the flow, that is, a return to the limit cycle. If, however, the frequency of the limit cycle is reduced (the amplitudes are reduced), the plate starts to acquire energy from the flow, that is, there is again a return to the limit cycle.

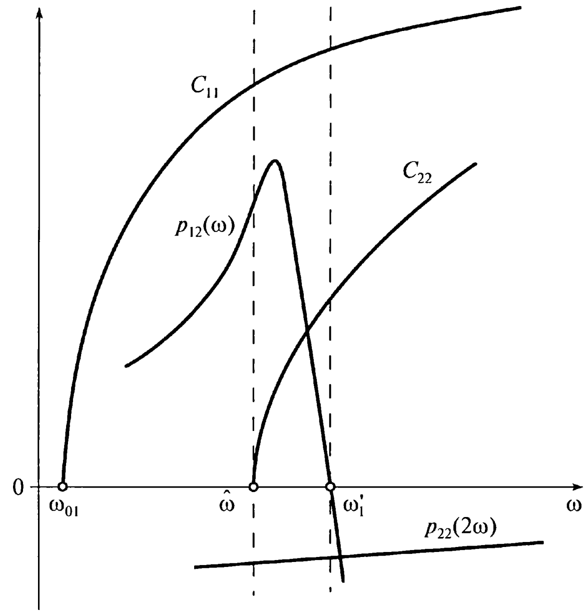

Fig. 3.

We emphasize that the stability of the resonance limit cycle is caused by the fact that, by virtue of the stability of parametric resonance, the amplitudes of the two harmonics are rigidly connected to the frequency by Eq. (5.2) and they cannot be separately excited. In the case of second mode single-frequency vibrations, the growth of the perturbation through the first mode (Section 4) was due to the fact that it is generally possible to excite it separately from the second mode through which vibrations are already occurring.

In the proof of stability that has been presented, only perturbations of the frequencies and amplitudes were considered while preserving the periodicity of the motion. The stability with respect to an arbitrary small perturbation requires a considerably more complex analysis, due to the fact that, in the case of an arbitrary non-periodic perturbation, it is no longer possible to reduce the unsteady gas pressure to aerodynamic damping and, in general, to any differential relation between the perturbation of the gas pressure and the bending of the plate. However, in the case of a tensioned extended plate under conditions of single-mode flutter (that is, of a weak effect of the flow on the plate dynamics), the initial stage in the development of instability of the limit cycle apparently always only shows up in the form of a gradual evolution of the limit cycle, that is equivalent to the small change in the frequencies and amplitudes considered.

We will now estimate how this limit cycle evolves when there is a further increase in the Mach number. In the case of pinning

$$a_{11} \approx 6.979/L^{5/2}, \quad a_{22} \approx 27.915/L^{5/2}$$

and, in the case of clamping

$$a_{11} \approx 8.699/L^{5/2}, \quad a_{22} \approx 32.587/L^{5/2}$$

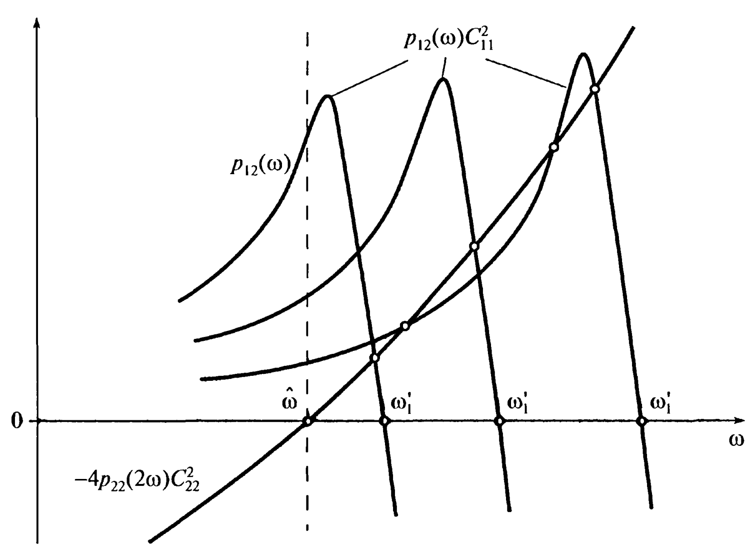

Fig. 4.

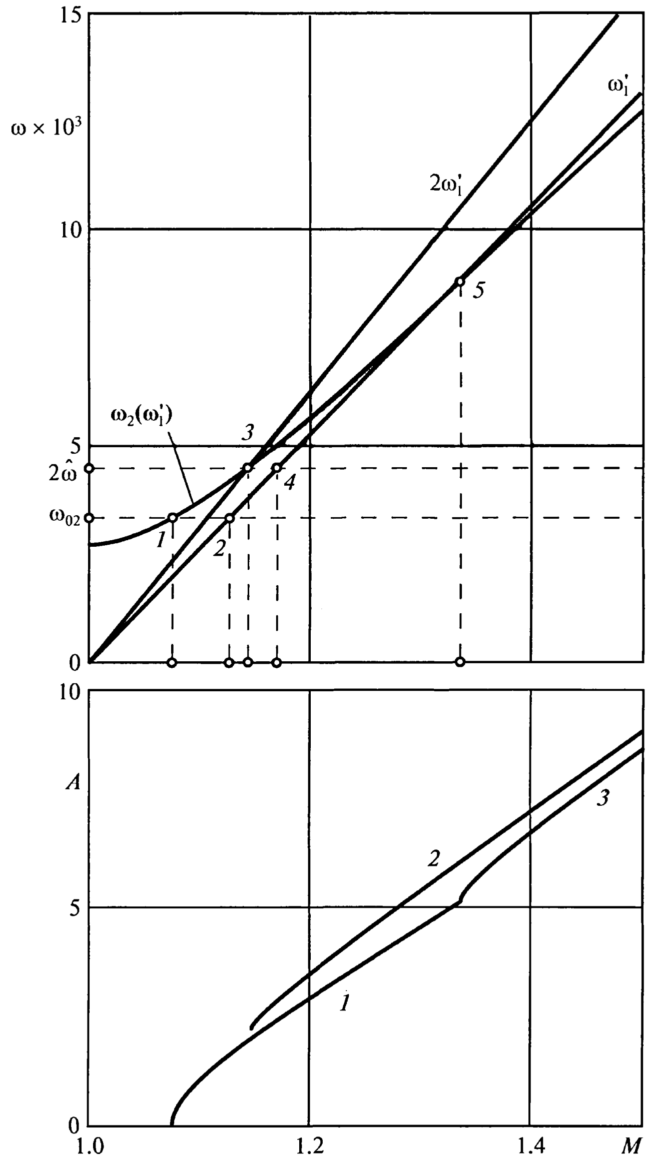

Fig. 5.

In both cases (and, consequently, also for "intermediate" types of boundary conditions), the coefficient of ω^2 in the second equality of (5.2) is greater than in the first. Hence, when the frequency increases after the onset of resonance, the amplitude of the second mode increases more rapidly than the amplitude of the first mode. At the same time, the value of $p_{12}(\hat{\omega})$ as M increases initially sharply increases and then decreases (see Fig. 2) while the quantity $p_{22}(2\hat{\omega})$ remains roughly constant. Hence, after the onset of resonance, there is one stable limit cycle, and the frequency lies in the section $\left[\hat{\omega}, \omega_1^{'}\right]$ nearer to the right end $\omega_1^{'}$ (Fig. 4). When M is sufficiently increased, two further resonance limit cycles appear since two additional points of intersection of the graphs in Fig. 4 arise. At the same time, the cycle corresponding to the middle point of intersection is unstable and there are therefore two stable resonance limit cycles. The frequency of the "new" limit cycle lies closer to the left end of the segment $\hat{\omega}$ and it corresponds to a lower frequency and smaller amplitude. The initial limit cycle does not vanish for small μ since the maximum of $p_{22} \sim \mu^{2/3}$ and $p_{22} \sim \mu$, that is, the maximum of $p_{12}(\omega)C_{11}^2$ in Fig. 4 is always greater than $-p_{22}(\omega)C_{22}^2$. This occurs until the frequency $\omega_2^{'}(M)$ reaches the value $2\hat{\omega}$. When $\omega_2^{'}(M) > 2\hat{\omega}$, the frequency of the weaker resonance cycle (if it has succeeded in appearing) increases: as previously, the left-hand side of Eq. (5.6) is negative when $\omega = \omega_1^{'}$ but already positive not only when $\omega = \hat{\omega}$ but also when $\omega = \omega_2^{'}/2$, since $\omega_2 < 2\omega_1$. The frequency of the "weak" resonance cycle therefore lies in the interval $\omega_2^{'} < \omega < \omega_1^{'}$.

We will now summarize the results obtained. The arrangement of the curves describing the vibrations conforming to the second mode is shown in the upper part of Fig. 5, the frequency of the non-resonance limit cycle $\omega_2^{'}(M)$, the doubled frequency of the non-resonance cycle of the fist mode $2\omega_1^{'}(M)$ and the frequency of the small vibrations conforming to the second mode, in the case of the developed non-resonance vibrations conforming to the first mode $\omega_2(\omega_1^1(M))$ (the solution of the first equation of (4.2) for $j = 2$ when $C_{21} = 0$). The plate was clamped and the values of the parameters (4.6) and $M_w = 0$, which corresponds to the absence of tension in the plate, were taken. When $1 < M < 1.08$, the plate is stable. When $M \approx 1.08$ (point 1), the first mode becomes growing and a non-resonance limit cycle arises with a vibration frequency $\omega_1^{'}(M)$. When $M \approx 1.13$ (point 2) the second mode becomes growing, but a stable limit cycle corresponding to it does not arise (Section 4). When $M \approx 1.14$ (point 3), the frequency $\omega_1'(M)$ is equal to half the frequency of the small vibrations conforming to the second mode and a second limit cycle arises, that is, a 1:2 resonance limit cycle. Starting from $M \approx 1.17$ (point 4), the frequency and amplitude of the second resonance cycle (if it has succeeded in emerging) start to grow. When $M \approx 1.34$ (point 5), the non-resonance cycle conforming to the first mode becomes unstable and a non-resonance two-frequency limit cycle conforming to the two first modes emerges instead of it. Starting from this instant, there are two stable limit cycles consisting of two modes, a non-resonance cycle and a resonance cycle.

In a unique case, that is, the case of pinning when $\omega_2'(M) = 2\omega_1'(M)$, points 3, 4, and 5 coincide. In this case, the resonance limit cycle coincides with the non-resonance limit cycle.

The amplitudes of the corresponding limit cycles divided by the thickness of the plate $A = \sqrt{2/L}(C_{11} + C_{22})$ are shown for a clamped plate in the lower part of Fig. 5. Curve *1* is for the single-frequency cycle, *2* is for the 1:2 resonance cycle and *3* is for the non-resonance two-frequency cycle. It is seen that the resonance limit cycle has the greatest amplitude.

6. Multifrequency vibrations with an odd internal resonance

Unlike an even resonance, as analytical study, even in the case of the simplest 1:3 resonance, meets with difficulties, since the equations for the amplitudes of the fundaanmental harmonics contain a phase shift that must be determined from equations containing aerodynamic amplification. A system of four non-separable non-linear algebraic equations is obtained, the solutions of which behave in a rather complex manner. On account of this, the behaviour of the limit cycle is will henceforth be studied qualitatively.

To be specific, we shall assume that the first and third modes enter into the resonance. We will seek a solution in the form

$$A_1 = C_{11}\cos\omega t + C_{13}\cos(3\omega t + \varphi) + \ldots$$
$$A_3 = C_{31}\cos(\omega t + \alpha) + C_{33}\cos(3\omega t + \beta) + \ldots \quad (6.1)$$

The system of equations is obtained from system (4.1) by replacing the index 2 by 3. We shall assume that, under the action of the flow, the plate executes quasiharmonic vibrations conforming to the first mode, $C_{13} \ll C_{11}$, which excite resonance vibrations of a third mode with a trebled frequency, $C_3 \ll C_{33}$.

Substituting (6.1) into the system of equations, we write out the cosine equations, that is, the equations for the amplitudes, and the sine equations, that is, the equations for the phase shifts. Only the leading terms are retained in the equations for the fundamental amplitudes C_{11} and C_{33} (obtained from the relations for $\cos\omega t$ in the equation for A_1 and for $\cos(3\omega t + \beta)$ in the equation for A_2). We obtain

$$C_{11}^3 + \frac{2}{3}\eta_1 C_{33}^2 C_{11} + \xi_1 C_{33} C_{11}^2 \cos\beta = \frac{4(\omega^2 - \omega_{01}^2)}{3\kappa_1} C_{11}$$
$$\frac{2}{3}\eta_3 C_{11}^2 C_{33} + C_{33}^3 + \frac{1}{3}\xi_3 C_{11}^3 \cos\beta = \frac{4(9\omega^2 - \omega_{03}^2)}{3\kappa_3} C_{33} \quad (6.2)$$

Here,

$$\eta_j = (a_{11}a_{33} + 2a_{13}^2)/a_{jj}^2, \quad \xi_j = a_{11}a_{13}/a_{jj}^2, \quad \kappa_j = Ka_{jj}^2$$

Unlike solution (5.2), terms with $\cos\beta$, that are now important in the first approximation, have appeared in the equations. The equations for the subharmonics express C_{13} and C_{31} in terms of the fundamental amplitudes; they are no longer required.

We now turn to the equations obtained by equating the coefficients on the sines of the fundamental harmonics, retaining only the leading terms:

$$2p_{12}(\omega)\omega = \frac{3}{4}\kappa_1\xi_1 C_{11}C_{33}\sin\beta \quad (6.3)$$

$$6p_{32}(3\omega)\omega C_{33} = -\frac{1}{4}\kappa_1\xi_1 C_{11}^3\sin\beta \quad (6.4)$$

whence, in particular, it follows that

$$p_{12}(\omega)C_{11}^2 + 9p_{32}(3\omega)C_{33}^2 = 0 \quad (6.5)$$

which expresses the following: the amount of energy transferred from the flow to the plate via the first mode after one period of the vibrations is equal to the amount of energy transferred by the plate back into the flow via the third mode. The overall work of the flow on the plate after a period of the vibrations is equal to zero by virtue of equality (4.7).

The equations following from equating the coefficients on $\sin(3\omega t + \varphi)$ in the equation for A_1 and $\sin(\omega t + \alpha)$ in the equation for A_3 serve to determine the phase shifts α and φ and these equations always have a solution.

The main difficulty, as mentioned above, is the fact that system (6.2) contains the phase shift β, determined from Eq. (6.3). Numerical solutions of system (6.2) are shown in Fig. 6 for different values of $\cos\beta$ (in the case of clamped plate, $D = 19.78$, $M_w = 0$, $L = 300$). Suppose the frequency ω_1' exceeds the resonance frequency $\hat{\omega}$ corresponding to $\cos\beta = 0$ and system (6.2) has the solutions C_{11} and C_{33}. It is proved exactly the same way as the case of 1:2 resonance that Eq. (6.5) has a root ω, the vibration frequency of the resonance limit cycle. The phase β, that can be made equal to $\pm\pi/2$ by a suitable choice of μ, is then determined from (6.3). The solution "makes itself self-consistent" in the sense that the set of quantities C_{11}, C_{33}, ω, $\beta = \pi/2$ obtained for a chosen value of μ satisfies system of equations (6.2), (6.3) and (6.5).

We now fix this value of μ and, by increasing the Mach number M, we somewhat change the frequency of the cycle and, correspondingly, the amplitudes. Near the resonance frequency $\hat{\omega}$, the rate of change in $C_{33}(\omega)$ is infinite (since $C_{33}(\omega) \sim \sqrt{\omega - \hat{\omega}}$) and, consequently, the

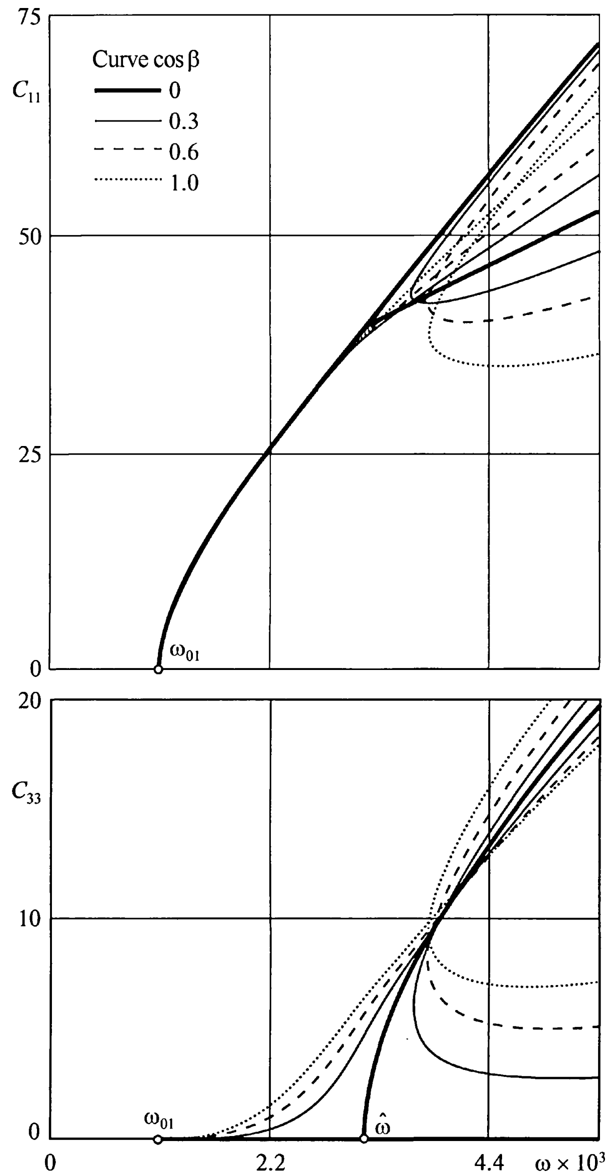

Fig. 6.

rate of change of the right-hand side of (6.3) when sinβ = 1 is also infinite while the left-hand side is finite. We obtain that, when β = ±π/2, Eq. (6.3) cannot be satisfied and, consequently, its solution $\beta \neq \pi/2$. It is seen in Fig. 6 that, in this case, three resonance solutions are formed instead of one. However, they cannot all always satisfy Eqs (6.3) and (6.5).

A numerical investigation of system (6.2), (6.3), (6.5) shows that, depending on the values of the parameters μ and *M*, an increase in *M* can lead both to a transition of the only resonance solution onto the branch in Fig. 6, that has the smallest amplitude (the resonance limit cycle remains unique), as well as to the simultaneous formation of three resonance limit cycles.

7. Conclusion

It has been shown that the non-linear vibrations of a plate in the case of single-mode flutter can have different limit cycles, that is, non-resonance limit cycles for which the vibrations conforming to each mode occur such that there is no energy exchange with the flow and resonance limit cycles for which energy is transferred from the flow to a mode of the plate and then, through non-linear coupling, this energy is transferred to another mode that then transfers it back into the flow.

In the case of instability in one mode, there is a unique stable limit cycle and its frequency is determined by the condition that the work performed by the pressure of the gas over a vibrational period is equal to zero. Its amplitude is calculated using the frequency from the usual equations for the non-linear vibrations of a plate in a vacuum, that is, the single-mode flutter vibrations of a plate in a flow can be represented as free non-linear vibrations in a vacuum with the specified frequency of the linearly growing mode. The explicit dependence of the amplitude and the frequency of the limit cycle on the parameters of the problem is obtained.

In the case of an increase in the flow velocity and the appearance of a second growing mode at a certain time, as previously the cycle consisting of just the first mode is the unique stable non-resonance limit cycle. A still greater velocity is required in order for a non-resonance limit cycle consisting of both modes to arise and, in this case, the cycle consisting of the one mode becomes unstable.

Other stable limit cycles can exist for the same problem parameters in which two or more modes are in internal resonance and exchange energy as described above. The existence of such a cycle has been proved using the example of an internal 1:2 resonance. In the case of a 1:3 resonance, it is more difficult to establish the domains of existence and stability of such a limit cycle. The vibration amplitude is greater in the case of a resonance limit cycle than in the case of non-resonance limit cycles. Resonance cycles are therefore more dangerous.

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