INFLUENCE OF THE BOUNDARY LAYER ON FLUTTER OF ELASTIC PLATE IN SUPERSONIC GAS FLOW

Vsevolod O. Bondarev, Vasily V. Vedeneev

Lomonosov Moscow State University
Faculty of Mechanics and Mathematics
Department of Hydromechanics
1 Leninskie Gory, Moscow, Russia
bondarev.vsevolod@yandex.ru, vasily@vedeneev.ru

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Abstract. Panel flutter is an aeroelastic instability of skin panels, which leads to excessive noise generation and their fatigue damage. Although linear stability of panels in uniform flow is studied in detail, the influence of the boundary layer is still an open question. Most studies of panel flutter in the presence of the boundary layer are devoted to $(1/7)^{th}$ power velocity law and yield stabilising effect of the boundary layer. Recently Vedeneev (*J. Fluid Mech.*, vol. 736, 2013, pp. 216–249) considered inviscid shear layer with arbitrary velocity and temperature law, and showed that for generalized convex boundary layer profile, the decrease of the growth rates of "supersonic" perturbations (responsible for single mode panel flutter) is accompanied by destabilisation of "subsonic" perturbations that are neutral in uniform flow. More surprising result obtained is that for the boundary layer profile with generalised inflection point the effect of the layer is destabilising even for "supersonic" perturbations. However, these results were obtained by using the long-wave approximation of the Rayleigh equation, which is not always applicable to growing waves.

To investigate short waves, in the first part of this study we solve the Rayleigh equation numerically and investigate the inviscid stability of short-wave perturbation of elastic plate in the presence of the boundary layer. We show that the short waves can have even larger growth rates than the long-wave approximation predicts.

In the second part of the work, the viscous disturbances are also taken into account, i.e. large finite Reynolds numbers are considered. We use asymptotic expansion for viscous and temperature solutions as $R \to \infty$ and investigate their influence on the unsteady pressure acting on the plate. It is shown that there exist boundary layer profiles so that finiteness of the Reynolds number yields additional increase of the growth rate comparing to both uniform flow and inviscid shear layer.

1 INTRODUCTION

Flutter of skin panels of flight vehicles is a dangerous phenomenon leading to high-amplitude vibrations of panels and their fatigue destruction. From theoretical point of view, the problem lies in instability of elastic plates or shells moving in air with supersonic speeds. As a rule, uniform air flow over the plate is considered, and the boundary layer is neglected. Intensive studies of this problem were conducted in the 1950-1970s and summarised in reviews [1–3]. During the last decade renewed interest to panel flutter has arisen due to new computational techniques devoted to nonlinear limit cycle oscillations [4], use of new materials for skin panels [5,6], and aerodynamic models suitable for transonic and low supersonic flight conditions [7–10].

While the stability of elastic panels in uniform flows is studied more or less in detail, effect of the boundary layer still remains an open question. The only experimental studies [11, 12] devoted specifically to the boundary layer influence on panel flutter, showed that it has stabilising effect, namely, critical dynamic pressure increases when the boundary layer is thicker. Theoretical studies [13–16], were devoted to panel flutter in the presence of the boundary layer with $(1/7)^{th}$ power velocity law, representing typical turbulent boundary layer profile. However, two common shortcomings of those papers can be pointed out. First, boundary layer profile for arbitrary flow conditions and panel locations can essentially differ from (1/7)th power law. Second, panel flutter can occur in two distinct form, coupled-mode and single-mode flutter [9]. They have different physical mechanisms so that their response to the presence of the boundary layer is also different. In the studies cited above flutter type was not distinguished so that it is not clear which one was examined for boundary layer effect.

In the recent study [17], an asymptotic analysis was performed for the problem of long plates in inviscid shear layer with supersonic mean flow, when the plate length $L \to \infty$ and Reynolds number $R \to \infty$. It was shown that the action of the boundary layer on coupled-mode flutter of a plate is as follows: it has positive effect in case of flutter, i.e. growth rate of the fluttering mode decreases, but it destabilises the plate in case of stability in uniform flow. The action of the boundary layer on single-mode flutter is more complex and depends on the boundary layer profile. Eigenmodes of the plate are split into two groups, supersonic and subsonic modes, which experience different behaviour due to presence of the boundary layer. Namely, for generalised convex boundary layer profiles, supersonic modes are stabilised (growth rates decrease), while subsonic are destabilised. For profiles with generalised inflection point, supersonic modes are destabilised for small boundary layer thickness δ and stabilised for large δ, while subsonic modes are damped.

For analysis of the problem [17], the global instability criterion [18] was used, which reduced the problem to the behaviour of waves in an infinite plate. Since the coupled-mode flutter of a finite-length plate is governed by long waves with wave number $k \sim \mu^{1/3}$, where μ is the ratio of the flow density to the plate material density and considered as a small parameter, the first term of Heisenberg expansion was used for solving Rayleigh equation. The same expansion was used for analysis of single mode flutter, which is governed by shorter waves, $k \gg \mu^{1/3}$. However, first-term Heisenberg expansion can be used only for $k \ll 1/\delta$, so that the results obtained for single mode flutter are valid only for small boundary layer thickness δ.

In this paper we re-examine the action of the boundary layer on growing waves and remove the limit of small δ by using numerical solutions of Rayleigh equation. In the section 2 we describe the formulation of the problem. In section 3 we discuss the closed-form solution [17] and its limitation, and describe numerical method for solving Rayleigh equation and dispersion relation. Section 4 is devoted to inviscid analysis of waves in infinite plates and comparison of growth rates obtained through the first-term Heisenberg expansion and the numerical solution of the full Rayleigh equation. Finally, in section 5 we investigate the effect of finite Reynolds numbers by using the WKB-expansion of viscous solutions for $R \gg 1$.

2 FORMULATION OF THE INVISCID PROBLEM

We consider the stability of elastic plate in a shear gas flow (figure 1). The flow represents the boundary layer over the plate surface, and its undisturbed velocity and temperature profiles, $u_0(z)$ and $T_0(z)$, respectively, are given. The problem is investigated in 2-D formulation; also,

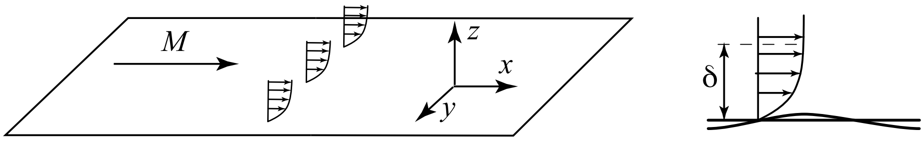

Figure 1: Gas flow over elastic plate.

we neglect the growth of the boundary layer so that the unperturbed flow does not depend on x. All variables are assumed to be non-dimensional, with the speed of sound and temperature of the flow outside the boundary layer taken as the velocity and temperature scales, the plate thickness as the length scale, and plate material density as the density scale.

The plate motion is governed by Kirchhoff-Love small deflection plate theory. In a dimensionless form, the plate equation is as follows:

$$D\frac{\partial^4 w}{\partial x^4} - M_w^2 \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial t^2} + p(x,0,t) = 0, \tag{1}$$

where $w(x,t)$ is the plate deflection, D is the dimensionless plate stiffness, M_w is the square root of the dimensionless in-plane tension force, and $p(x,z,t)$ is the flow pressure disturbance induced by the plate motion, such that p is a function of w.

Let us consider plate of infinite length. Since neither plate, nor flow properties depend on x, this admits the perturbations of a travelling-wave type that govern the stability of the system. Let $w(x,t) = e^{i(kx-\omega t)}$ and $p(x,z,t) = p(z)e^{i(kx-\omega t)}$ be the perturbations of the plate deflection and the flow pressure (note that due to linearity of the stability problem we may assume the deflection amplitude to be unit). Substitution into (1) yields the dispersion relation

$$Dk^4 + M_w^2 k^2 - \omega^2 + p(0) = 0. \tag{2}$$

Since the pressure $p(0)$ is induced by the plate, we need to express it through the plate deflection to obtain the dispersion relation in a self-sufficient form.

The flow is assumed to be laminar; the Reynolds number $R \to \infty$. This means that small perturbations of the flow are governed by the inviscid Rayleigh equation. Let $v(z)e^{i(kx-\omega t)}$ be the perturbation of vertical flow velocity component. Then the compressible Rayleigh equation [19] takes the following form:

$$\frac{d}{dz}\left(\frac{(u_0-c)dv/dz - v du_0/dz}{T_0 - (u_0-c)^2}\right) - \frac{1}{T_0}k^2(u_0-c)v = 0, \tag{3}$$

where $c = \omega/k$ is the phase speed of the wave. When $v(z)$ is found, amplitude of pressure perturbation is expressed through $v(z)$ as follows:

$$p(z) = \frac{\mu}{ik}\frac{(u_0-c)dv/dz - v du_0/dz}{T_0 - (u_0-c)^2}, \tag{4}$$

where μ is the dimensionless density of the flow outside the boundary layer. We will assume that μ is a small parameter, as for all gas properties, flow conditions, and plate materials available in applications it has an order of 0.001 or less.

The boundary conditions for the Rayleigh equation are as follows. First, at the plate surface $z = 0$ impenetrability condition must be satisfied. Since the plate shape is $w(x) = e^{i(kx-\omega t)}$, we obtain:

$$v(0) = \frac{\partial w}{\partial t} + u_0(0)\frac{\partial w}{\partial x} = -i\omega, \quad z = 0. \tag{5}$$

Second, radiation condition must be satisfied as $z \to +\infty$. We will assume that the flow profile outside the boundary layer is constant, i.e. $u_0 \equiv M$, and $T_0 \equiv 1$ for $z > \delta$, where M is the free-stream Mach number. This admits transferring of the radition condition from infinity to $z = \delta$ in the following way. For $z > \delta$, Rayleigh equation (3) is an equation with constant coefficients. Its solution satisfying radiation condition is an exponent $v(z) = Ce^{\gamma z}$, $\gamma = -\sqrt{k^2 - (M_\infty k - \omega)^2}$, where the square root branch is chosen so that $\operatorname{Re}\gamma < 0$ for $\operatorname{Im}\omega \gg 1$. At $z = \delta$ this solution must be matched with the solution inside the boundary layer. Matching condition is

$$v(z) = Ce^{\gamma z}, \frac{dv(z)}{dz} = C\gamma e^{\gamma z}, \quad z = \delta,$$

which after excluding C transforms to

$$\frac{1}{v}\frac{dv}{dz} = \gamma, \quad z = \delta. \tag{6}$$

Thus to calculate the pressure $p(0)$ in (2) we have to solve the Rayleigh equation (3) with boundary conditions (5), (6), and then use the expression (4).

When $\mu = 0$, i.e. there is no flow, solutions of the dispersion relation (2) $\omega(k)$ are real and correspond to bending plate waves. If $\mu \neq 0$, but $\delta = 0$, i.e. the flow is uniform, the downstream-travelling wave is growing for $0 < \operatorname{Re} c < M - 1$, neutral for $M - 1 < \operatorname{Re} c < M + 1$, and damped for $\operatorname{Re} c > M + 1$ [20]. The thresholds

$$\operatorname{Re} c = M \pm 1 \quad \Longleftrightarrow \quad k_{M\pm1} = \sqrt{\frac{(M \pm 1)^2 - M_w^2}{D}} \tag{7}$$

correspond to waves travelling with speed of acoustic perturbations of the flow. Upstream travelling waves, $\operatorname{Re} c < 0$, are always damped.

It can be proved that the waves with $\operatorname{Re} c > M$ and $\operatorname{Re} c < 0$ in the presence of the boundary layer remain either neutral, or damped. That is why we will restrict ourselves to the influence of the shear layer on the waves with $0 < \operatorname{Re} c < M$.

3 METHODS FOR SOLVING RAYLEIGH EQUATION AND DISPERSION RELATION

We will use three approaches for solving Rayleigh equation, finding $p(0)$, and then solving the dispersion relation (2). The first approach, which we will refer to as "analytical", was used in [17]. Though it admits general investigation of the wave behaviour, it is not valid for short waves. The second approach, which we will refer to as "numerical", consists in numerical solution of both Rayleigh equation and the dispersion relation. Obviously, it is valid for arbitrary wave lengths; we will use it for analysis of short wave begaviour. The third, intermediate approach, which we will call "semi-analytical", consists in analytical solution of Rayleigh equation (similar to "analytical" approach), but numerical solution of the dispersion relation. This approach will be used as a connection between analytical and numerical results.

3.1 Analytical solution for "short" waves

Solution of the Rayleigh equation can be written in form of a series in k^2, known as Heisenberg expansion [21]. Consider the first term of this expansion, which is valid for $k \ll 1/\delta$, i.e. for the wave lengths that are much longer then the boundary layer thickness. Such consideration is equivalent to neglecting the second term of the order of k^2 in (3). When this term is omitted,

(3) is solved in a closed form. Satisfying boundary conditions (5), (6), we obtain the pressure in the form [17]:

$$p(0) = -\mu \left(\left(\frac{(Mk-\omega)^2}{\sqrt{k^2-(Mk-\omega)^2}} \right)^{-1} + \delta \left(\int_0^1 \frac{T_0(\eta)d\eta}{(u_0(\eta)-c)^2} - 1 \right) \right)^{-1},$$

where $\eta = z/\delta$ is inner boundary layer coordinate, and $c = \omega/k$ is the phase speed. Here the first term in parentheses represents the contribution of the uniform flow outside the boundary layer, while the second term represents the contribution of the boundary layer. Substitution in (2) yields the dispersion relation

$$\mathcal{D}(k,\omega) = (Dk^4 + M_w^2 k^2 - \omega^2) - \mu \left(\left(\frac{(Mk-\omega)^2}{\sqrt{k^2-(Mk-\omega)^2}} \right)^{-1} + \delta \left(\int_0^1 \frac{T_0(\eta)d\eta}{(u_0(\eta)-c)^2} - 1 \right) \right)^{-1} = 0. \quad (8)$$

As $\delta \to 0$, the dispersion relation (8) coincides with the dispersion relation for a plate in uniform flow [20, 22].

Closed-form solution of the dispersion relation (8) can be obtained if an additional condition is satisfied: wave number $k \gg \mu^{1/3}$, i.e. the wave is not too long. This condition means that the order of the first term in (8) is larger then the second, i.e. the wave is mostly governed by the plate, while the flow gives just a small correction of the order of μ. Under these conditions, the Taylor expansion in μ gives:

$$\omega(k,\mu) = \omega(k,0) - \frac{\mu}{2\omega(k,0)} \left(\left(\frac{(M_\infty k-\omega)^2}{\sqrt{k^2-(M_\infty k-\omega)^2}} \right)^{-1} + \delta \left(\int_0^1 \frac{T_0(\eta)d\eta}{(u_0(\eta)-c)^2} - 1 \right) \right)^{-1}, \quad (9)$$

where the expression in parentheses is calculated at $\mu = 0$. Here the first term $\omega(k,0)$ is a frequency of the plate in vacuum, while the second term of the order of μ is a small frequency correction due to the flow.

Note that for solution (9) both conditions $k \gg \mu^{1/3}$ and $k \ll 1/\delta$ must be satisfied, i.e. the wave lengths are bounded both above and below. Such waves were called "short" (in quotation marks) [17], since they do not include shorter wave lengths, $k \gtrsim 1/\delta$. Let us consider numerical methods for solving Rayleigh equation and the dispersion relation, which are valid for waves of arbitrary length.

3.2 Numerical solution for arbitrary wave lengths

In order to overcome the restriction $k \ll 1/\delta$, and consider arbitrary wave lengths, we solve the Rayleigh equation numerically through the Runge-Kutta method. The difficulty associated with logarithmic singularity at the critical point is eliminated by solving the equation along a path in the complex z-plane. Namely, the algorithm is as follows:

1. For a given velocity profile $u_0(z)$ we find the critical point z_c as a root of the equation $u_0(z) = c$.

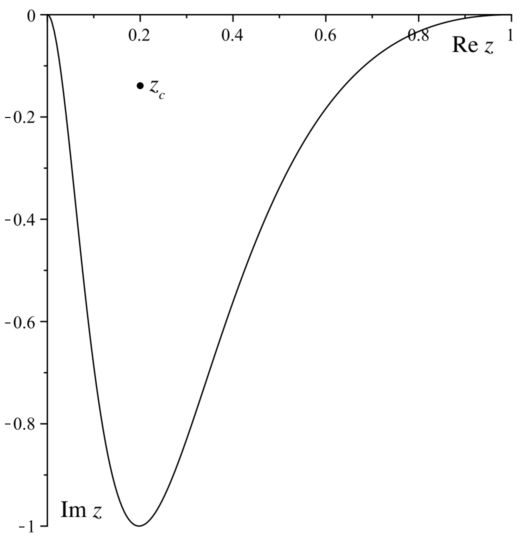

Figure 2: Integration path chosen for solving the Rayleigh equation for the velocity profile $u_0(\eta) = M\sin(\pi\eta/2)$, $M = 1.6$, $c = 0.5 - 0.33i$. The critical point is $z_c \approx 0.20 - 0.14i$, $\delta = 1$.

2. If the critical point exists, i.e. $0 < \operatorname{Re} z_c < \delta$, then according to Lin's rule, this point must be passed below in the complex z-plane in order to obtain solution which is a limit of viscous solution as $R \to \infty$ (vanishing viscosity) [21]. In order to enforce this rule, we choose a smooth path passing below the critical point in the complex z-plane. If $\operatorname{Im} z_c$ is positive and not too close to zero (namely, $\operatorname{Im} z_c \geq \delta/2$), then the path is a real segment $[0; \delta]$. Otherwise, the path is complex, as shown in figure 2. Note that for small positive $\operatorname{Im} z_c$ the equation can be integrated along the real segment, but we still use the complex path to avoid numerical difficulties due to closeness of the critical point, which is a point of singularity for the Rayleigh equation.

3. Boundary value problem is reduced to two initial value problems by the standard shooting method. They both are solved along the chosen path through the Runge-Kutta method. The velocity perturbation $v(z)$ and its derivative $dv(z)/dz$ are found.

4. Finally, by using formula (4), we calculate unsteady pressure $p(0)$ on the plate surface.

For solving the dispersion equation (2), we use the following iterative procedure. Let the wave number k be given, and we search $\omega(k)$ satisfying the dispersion equation. In the first step, ω_1 equals to the natural frequency of the plate in vacuum:

$$\omega_1 = \sqrt{Dk^4 + M_w^2 k^2}.$$

Now, let us have n-th iteration, ω_n. We numerically solve the Rayleigh equation and calculate the unsteady pressure $p(0, \omega_n)$ according to the procedure described above. Then we put

$$\omega_{n+1} = \sqrt{Dk^4 + M_w^2 k^2 + p(0, \omega_n)}.$$

Iterations are repeated until the desired accuracy is achieved, i.e., $|\mathcal{D}(k, \omega_n)| < \varepsilon$.

Convergence study shows that $N = 3000$ points is enough for discretizing Rayleigh equation along the integration path to get accurate solution through Runge-Kutta method. For the iterative solution of the dispersion relation, $\varepsilon = 10^{-10}$ gives well-converged solution $\omega(k)$ for all wave numbers k considered in examples below.

3.3 Semi-analytical solution for "short" and long waves

The analytical solution is not applicable in the case of very long waves, $k \sim \mu^{1/3}$, since expressions in the first and the second parentheses in (8) have the same order, and Taylor's expansion (9) is not valid. In this paper we are interested in the role of the second term of the Rayleigh equation, which is essential for $k \gtrsim 1/\delta$. However, depending on particular boundary layer profile, the latter segment can overlap the range $k \sim \mu^{1/3}$, i.e. there can be no range of wave numbers where analytical solution is valid. In this case, instead of analytical, we use "semi-analytical" approach, which consists in the following. We use the dispersion relation (8), i.e. neglect the second term in the Rayleigh equation. However, this relation is solved numerically in the same manner as in the numerical approach. Frequencies $\omega(k)$ obtained through semi-analytical approach are valid for $k \ll 1/\delta$, without any restriction for small k. Comparison of semi-analytical and numerical results will show the role of the second term in the Rayleigh equation for very short waves, $k \gtrsim 1/\delta$.

4 RESULTS: INVISCID PERTURBATION OF THE BOUNDARY LAYER

4.1 Generalised convex boundary layer profiles

We will call the boundary layer profile with $(u_0'/T_0)' > 0$ for $z \in [0; \delta]$ the generalised convex profile. Analysing solution (9), it was proved [17] that for such profiles the action of the boundary layer on the wave is as follows:

1. If the wave is growing for $\delta = 0$ (i.e. $0 < \mathrm{Re}\, c < M - 1$), then it stays growing for $\delta \neq 0$, but the growth rate monotonically decreases toward zero when δ increases.

2. If the wave is neutral for $\delta = 0$ (i.e. $M - 1 < \mathrm{Re}\, c < M$), then for $\delta \neq 0$ it becomes growing. When δ increases, the growth rate increases, achieve its maximum at $\delta = \delta_m$, and then monotonically decreases toward zero.

These results were obtained with the second term in the Rayleigh equation neglected, i.e. very short waves were not considered. Now let us investigate the influence of this term.

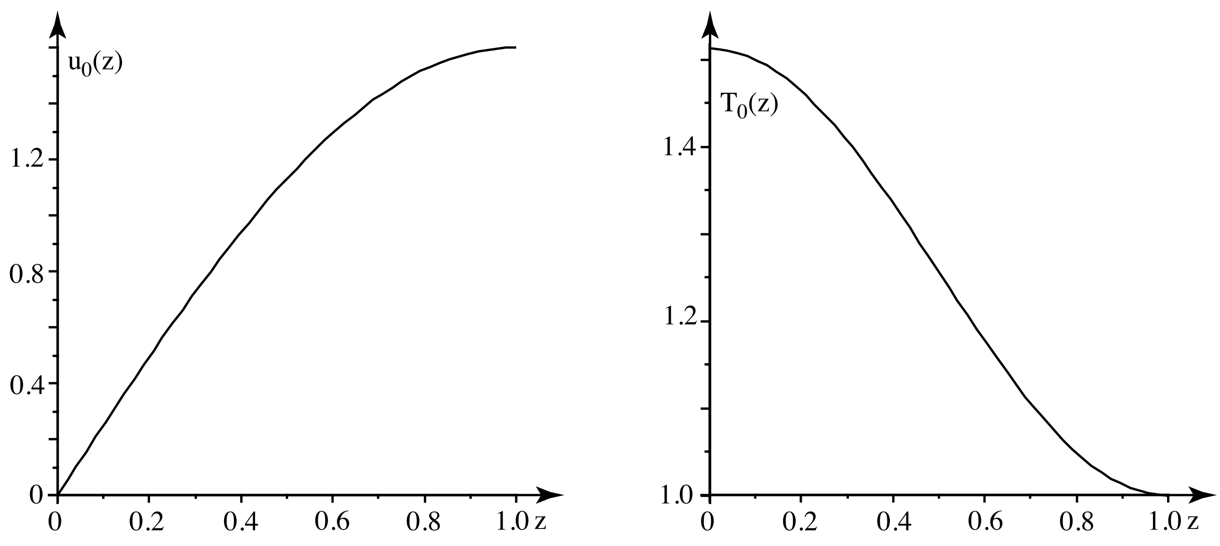

Figure 3: Stable generalised convex boundary layer profile (10), (12).

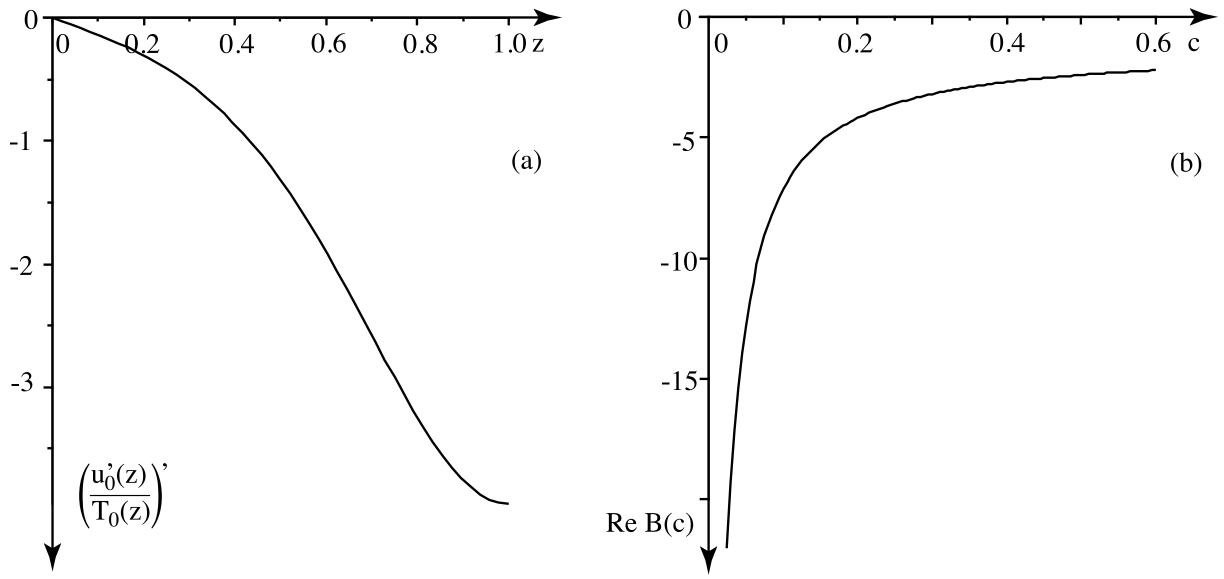

Figure 4: Generalised curvature (a); $\operatorname{Re} B(c)$ (b) for the boundary layer profile (10), (12).

Consider as an example the velocity profile

$$u_0(z) = M \sin\left(\frac{\pi}{2}\frac{z}{\delta}\right) \tag{10}$$

for parameters $M = 1.6$ and

$$D = 23.9, \quad M_w = 0, \quad \mu = 0.00012, \quad \gamma = 1.4. \tag{11}$$

For simplicity, in all examples hereunder we will assume that Prandtl number $Pr = 1$ and the plate is heat-insulated so that the temperature profile $T_0(u_0)$ is given using the same expression as in an adiabatic flow [23]:

$$T_0(u_0) = 1 + \frac{\gamma - 1}{2}(M^2 - u_0^2). \tag{12}$$

The boundary layer profile (10), (12) is shown in figure 3.

First, let us show that this boundary layer is stable itself (i.e., over the rigid plate) in inviscid approximation. Indeed, the stability criterion of subsonic disturbances (i.e., disturbances with the phase speed $M - 1 < c$) is as follows [19]: the profile must be generalised convex in the subsonic part of the layer, i.e.

$$\left(\frac{u_0'(z)}{T_0(z)}\right)' < 0, \quad z > z_s, \tag{13}$$

where $u_0(z_s) = M - 1$. This function is plotted in the figure 4a; clearly, the stability condition is satisfied. Next, for supersonic long-wave disturbances [17] showed that the stability criterion is as follows:

$$\operatorname{Re} B(c) < 0, \quad 0 < c < M - 1, \tag{14}$$

where

$$B(c) = \int_0^1 \frac{T_0(\eta)d\eta}{(u_0(\eta) - c)^2} - 1.$$

It is seen from the figure 4b that the stability condition for supersonic disturbances is also satisfied, so that the boundary layer profile (10), (12) is indeed stable.

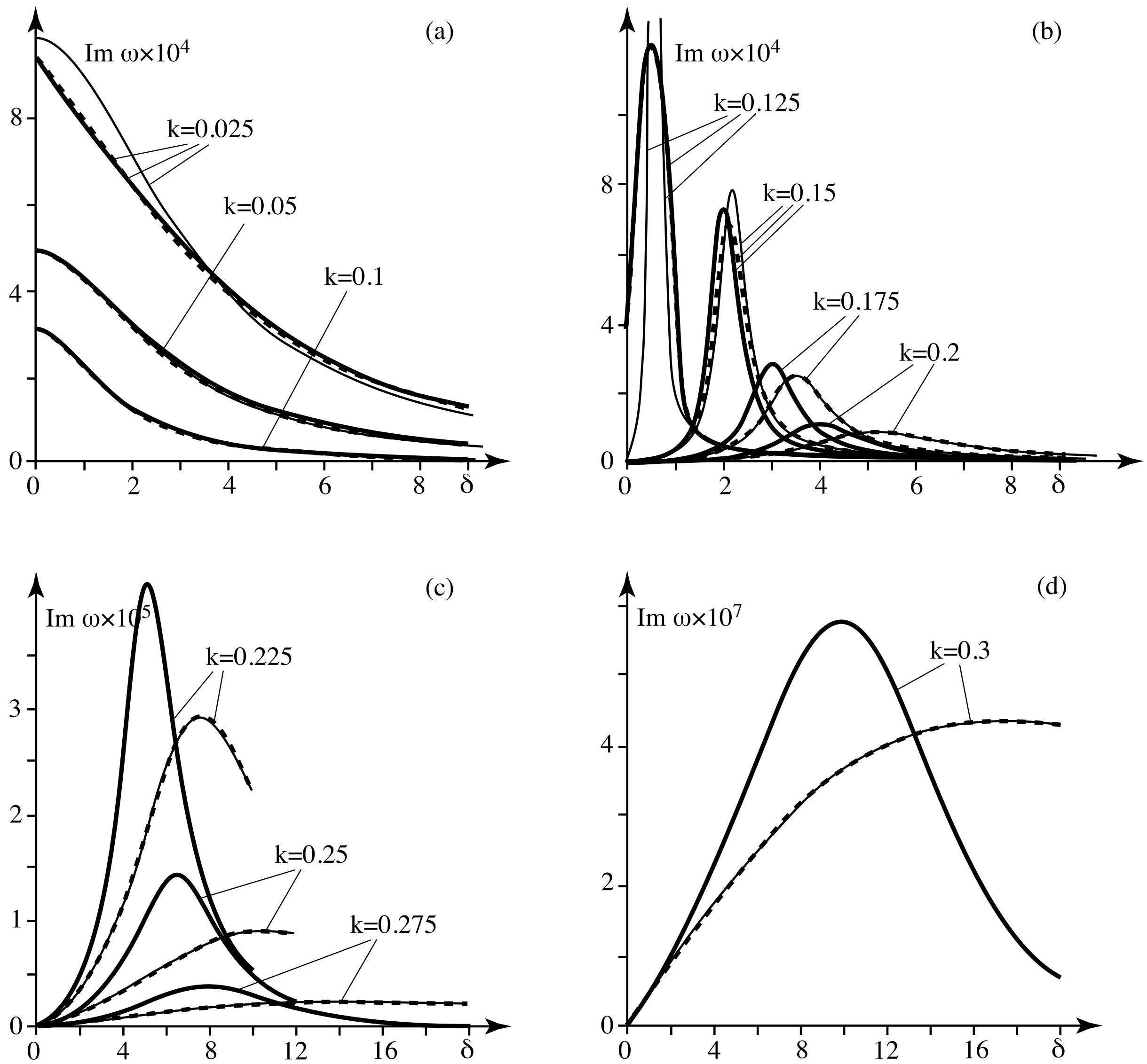

Figure 5: Growth rates $\mathrm{Im}\,\omega(\delta)$ for waves that are growing (a), neutral (b,c,d) in uniform flow. Boundary layer profile is given by (10), (12). Thin continuous, dashed, and thick continuous lines represent analytical, semi-analytical, and numerical solution, respectively.

Now let us consider the influence of the second term in Rayleigh equation on the travelling waves. Calculated growth rate versus the boundary layer thickness, $\mathrm{Im}\,\omega(\delta)$, are plotted for a range of wave numbers k in figure 5. For "moderate" wave lengths, $0.05 \leq k \leq 0.1$, there is an excellent agreement between the numerical, analytical and semi-analytical solutions. This range of wave numbers corresponds to waves growing in uniform flow, i.e. $k < k_{M-1} = 0.123$. For $k \leq 0.025$ analytical solution gives significant discrepancy with numerical and semi-analytical results, since the condition $k \gg \mu^{1/3}$ is not valid, and Taylor expansion (9) cannot be used. However, semi-analytical and numerical solutions, which do not use Taylor's expansion, are still in excellent agreement with each other.

For $k > 0.125$ significant difference between numerical and analytical solutions appears due to influence of the second term in Rayleigh equation. Results presented in figure 5 cover the full range of $k_{M-1} < k < k_M = 0.327$ and yield two corrections due to this term. First, actual maximal growth rates are higher than predicted by the analytical theory. The higher k, the higher relative increase of $\max \operatorname{Im} \omega(\delta)$ due to the second term. Second, the maximum growth rate is achieved at smaller boundary layer thickness.

We have checked several other generalised convex boundary layer profiles, including (10) with $M = 1.3$ and 2.0. In all cases the results are similar to plotted in figure 5, and the short wave effect of the second term in Rayleigh equation results in the increase of the growth rates in all cases where this effect is visible.

We conclude that the destabilisation of neutral waves with $M - 1 < c < M$ due to the boundary layer, initially predicted by analytical theory [17], is confirmed by the numerical solution of the full Rayleigh equation. Moreover, growth rates turn out to be higher than the analytical theory predicts.

4.2 Boundary layer profile with generalised inflection point

Now let us investigate boundary layer profile with generalised inflection point z_i, where $(u_0'/T_0)' = 0$. In subsonic flow such a profile would be unstable, since the existence of the generalised inflection point is necessary and sufficient condition for inviscid instability of subsonic disturbances [19]. However, in supersonic flow there exist such profiles that the generalised inflection point is located in supersonic part of the boundary layer, i.e. $u_0(z_i) < M - 1$. This means that, on one hand, subsonic disturbances are damped, since the criterion (13) is satisfied. On the other hand, supersonic disturbances can also be damped, since their stability condition (14) has no relation to the generalised inflection point. Consequently, such a profile can be stable and exist in reality.

For such boundary layer profiles, treating analytical solution, [17] proved that the action of the boundary layer on the wave is as follows:

1. If the wave is growing for $\delta = 0$ (i.e. $0 < \operatorname{Re} c < M - 1$), then it stays growing for $0 < \delta < \delta_1$, such that the growth rate is higher than in uniform flow. For thicker boundary layer, $\delta_1 < \delta < \delta_2$, the wave is still growing, but the growth rate is lower than in uniform flow. Finally, in thick boundary layers, $\delta > \delta_2$, the wave become damped.

2. If the wave is neutral for $\delta = 0$ (i.e. $M - 1 < \operatorname{Re} c < M$), then for $\delta \neq 0$ become growing. The behaviour of such waves is similar to that in generalised convex boundary layers.

Let us now consider numerical results. As an example, we use the following velocity profile:

$$u_0(z) = M\left(1 - \left(1 - \frac{z}{\delta}\right)^{2.4}\right) \times \cos\left(0.7\left(1 - \frac{z}{\delta}\right)^{7}\right)^{7} \tag{15}$$

for $M = 1.3$, parameters (11), and the temperature profile (12), which is shown in figure 6. Though the velocity profile (15) looks sophisticated, it just represents the function with one generalised inflection point located in supersonic part of the layer. Let us show that this boundary layer is stable in inviscid approximation. First, shown in figure 7a is the generalised curvature $(u_0'/T_0)'$. It is negative for $z > z_i = 0.133$, and $u_0(z_i) = 0.298 < 0.3 = M - 1$, i.e. the condition (13) is satisfied and subsonic disturbances are damped. Second, $\operatorname{Re} B(c)$ shown in figure 7b is negative for $c < M - 1$, i.e. the condition for supersonic disturbances (14) is also satisfied, and they are also damped.

As well as before, we calculated growth rates $\operatorname{Im} \omega(\delta)$ for a range of wave numbers. Results for $k < k_{M-1} = 0.061$, which correspond to growing waves in uniform flow, are shown in figure 8a. It is seen that there is a good agreement between semi-analytical and numerical

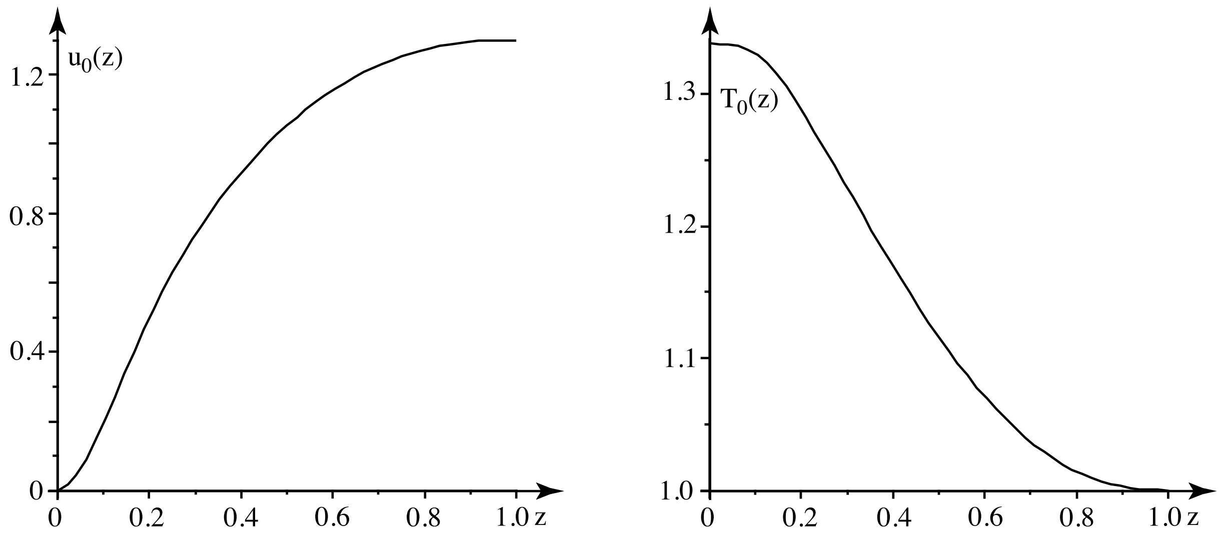

Figure 6: Stable boundary layer profile with generalised inflection point (15), (12).

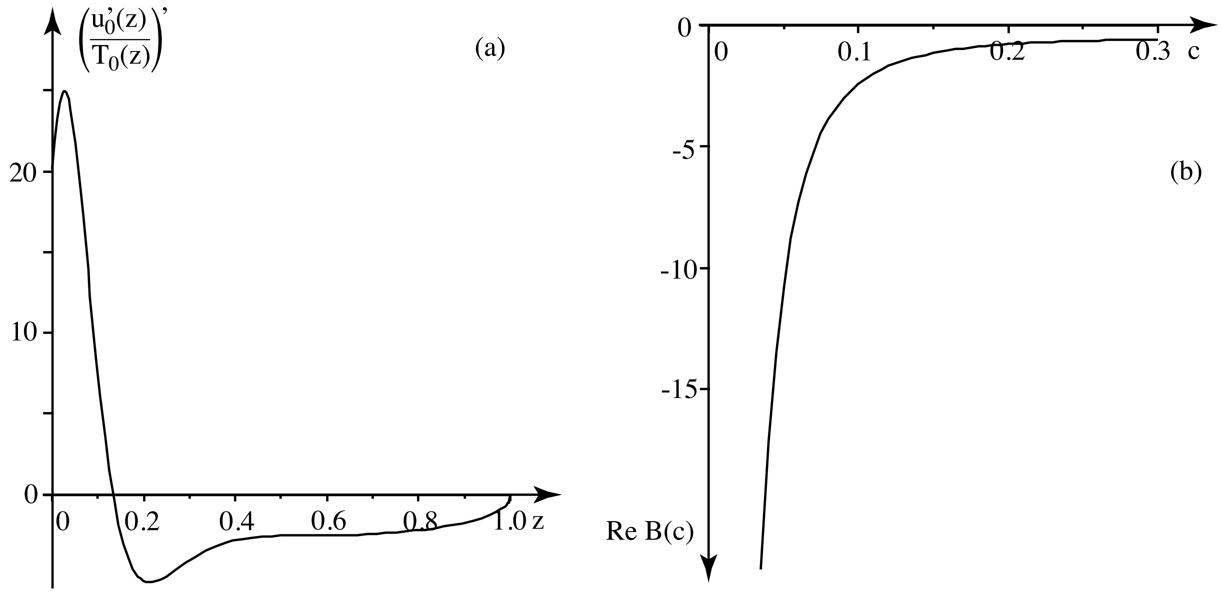

Figure 7: Generalised curvature (a); $\operatorname{Re} B(c)$ (b) for the boundary layer profile (15), (12).

results. For $k = 0.05$ analytical results are close, but for smaller k they are essentially different, since $k \sim \mu^{1/3}$, and Taylor's expansion is not applicable for such small k. It is seen that despite the short-wave effect is not pronounced, the growth rate obtained numerically is slightly higher than those obtained in semi-analytical approximation.

For $k_{M-1} < k < k_M = 0.266$ (figure 8b,c) the waves are neutral in uniform flow but become growing due to the boundary layer. Results for analytical and semi-analytical approximations coincide, since the wave number is high enough, and Taylor's expansion works well. However, numerical solution is significantly different, since the term of the order of k^2 in (3) becomes essential. It is clearly seen that for all k the actual growth rates are significantly higher than predicted by the analytical theory.

We conclude that as for generalised convex profiles, in all cases where the short-wave effect is visible, it results in higher growth rate than predicted analytically. In other words, the boundary layer is even more destabilising than the long-wave approximation predicts.

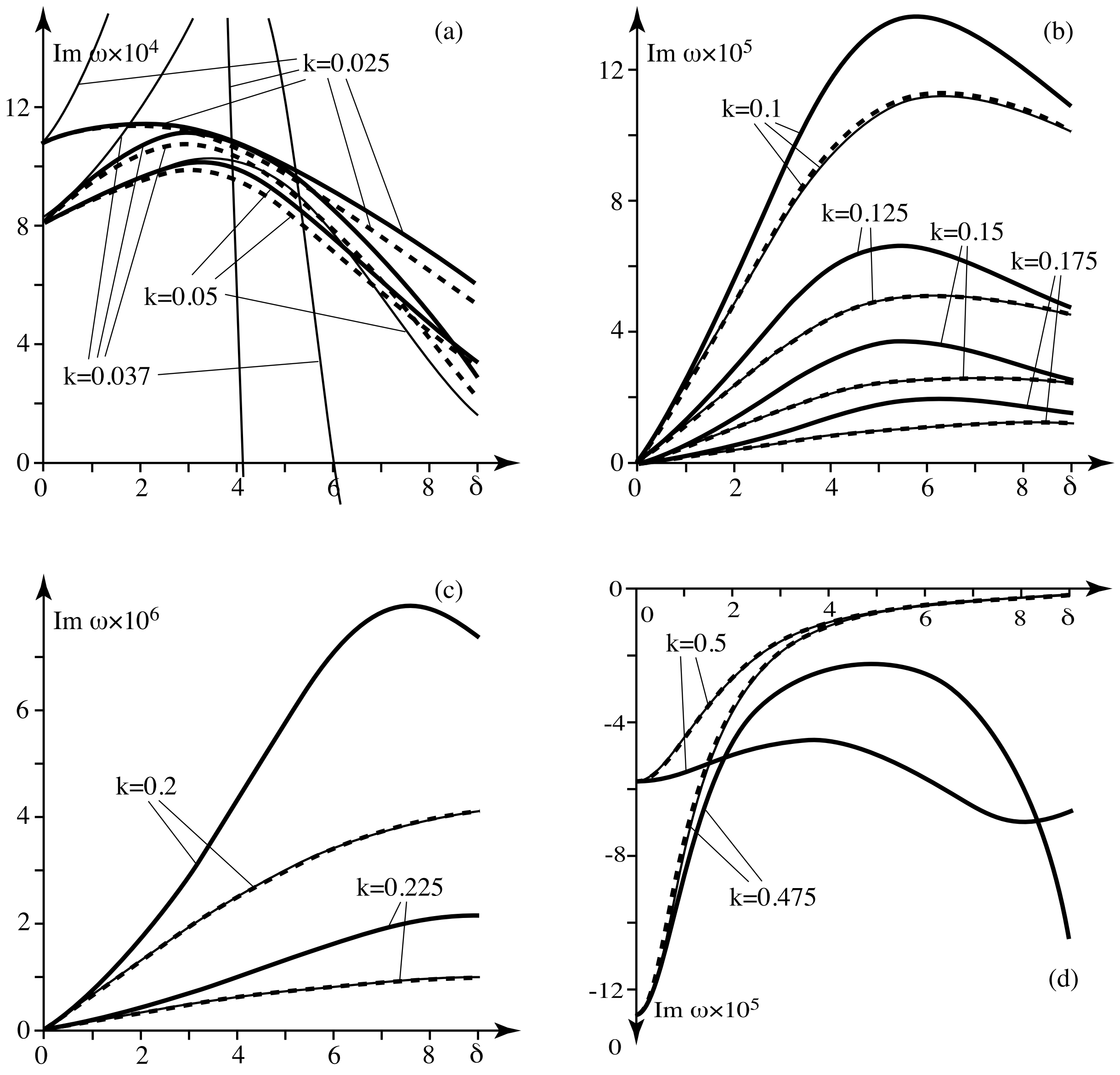

Figure 8: Growth rates $\mathrm{Im}\,\omega(\delta)$ for waves that are growing (a), neutral (b,c), damped (d) in uniform flow. Boundary layer profile is given by (15), (12). Thin continuous, dashed, and thick continuous lines represent analytical, semi-analytical, and numerical solution, respectively.

5 VISCOUS PERTURBATIONS OF THE BOUNDARY LAYER

In this section, we consider large Reynolds numbers and study the influence of the viscous perturbations on the growth rate. We assume that the Reynolds number is finite and investigate the first-term expansion of the growth rate in $\varepsilon = (kR)^{-1/2}$, which is considered a small parameter.

5.1 The first-order approximation in $(kR)^{-1/2}$ for the pressure perturbation

It is known [19] that the solution of system of equations for small perturbations of the boundary layer flow is the sum of three linearly independent solutions (regular, viscous and thermal), for which the radiation condition at infinity is satisfied:

$$\begin{cases} f(z) = c_1 f_r(z) + c_2 f_v(z) + c_3 f_t(z) \\ \phi(z) = c_1 \phi_r(z) + c_2 \phi_v(z) + c_3 \phi_t(z) \\ \theta(z) = c_1 \theta_r(z) + c_2 \theta_v(z) + c_3 \theta_t(z), \end{cases} \tag{16}$$

where f, ϕ, and θ are perturbations of the horizontal velocity, vertical velocity, and temperature of the flow. Indices r, v, t denote regular, viscous, and thermal solution, respectively. Constants c_j are found by satisfying the boundary conditions at the plate surface (no-slip condition and thermal insulation).

Regular solution tends to the solution of the inviscid Rayleigh equation as $\varepsilon \to 0$, and its expansion for small ε has the form:

$$f_r(z, \varepsilon) = f_{inv}^0(z) + f_r^2(z)\varepsilon^2 + \cdots, \tag{17}$$

Viscous and thermal solutions have the WKB-type asymptotic form as $\varepsilon \to 0$ [19]:

$$f(z, \varepsilon)_{v,t} = (f_{v,t}^0(z) + f_{v,t}^1(z)\varepsilon + f_{v,t}^2(z)\varepsilon^2 + \cdots) \exp\left(\frac{g_{v,t}(z)}{\varepsilon}\right).$$

Similarly, for other unknown functions, including the pressure perturbation, we have:

$$p(z) = c_1 p_r(z) + c_2 p_v(z) + c_3 p_t(z),$$

The calculation of WKB expansions for pressure perturbations in viscous and thermal solutions yields the following:

$$p_v(z) = O(\varepsilon^2), \quad p_t(z) = O(\varepsilon^2).$$

as $\varepsilon \to 0$. Using the expansion (17) for the regular solution p_r we obtain:

$$p(z, \varepsilon) = c_1(\varepsilon) p_{inv}(z) + O(\varepsilon^2). \tag{18}$$

Thus, to find the first term of the pressure expansion in ε, we need to calculate $c_1(\varepsilon)$, which is the only quantity representing the leading-order influence of the Reynolds number finiteness.

Consider the boundary conditions at the plate surface:

$$\begin{cases} f = 0, \quad z = 0 \\ \phi = -ic, \quad z = 0 \\ \theta' = 0, \quad z = 0 \quad \text{(thermally insulated plate)} \end{cases} \tag{19}$$

The substitution of (16) into (19) yields

$$\begin{cases} c_1 f_r(0) + c_2 f_v(0) + c_3 f_t(0) = 0 \\ c_1 \phi_r(0) + c_2 \phi_v(0) + c_3 \phi_t(0) = -ic \\ c_1 \theta_r'(0) + c_2 \theta_v'(0) + c_3 \theta_t'(0) = 0 \end{cases}$$

Using the Cramer's rule, the Rayleigh equations for the inviscid solution and explicit expressions for $f_{v,t}$, $\phi_{v,t}$, we find the expansion

$$c_1(\varepsilon) = c_{1inv} + c_1^1 \varepsilon + O(\varepsilon^2).$$

5.2 The case of long waves

To obtain the plate growth rates in a closed form, we will restrict ourselves by the case of "short" waves, i.e. such wave length that $k \ll 1/\delta$ (such that the Rayleigh equation can be solved analytically), but $k \gg \mu^{1/3}$, such that the Taylor expansion for ω can be used (Section 3).

Neglecting the term of power of k^2 in the Rayleigh equation, and calculating $c_1(\varepsilon)$, in the case $0 < c < M$ we obtain:

$$p(0) = -\mu \left(\left(\frac{(Mk-\omega)^2}{\sqrt{k^2-(Mk-\omega)^2}} \right)^{-1} + \left(\int_0^\delta \frac{T_0 dz}{(u_0-c)^2} - \delta \right) \right)^{-1} \times \left(1 - \varepsilon \left| \frac{\rho c}{\mu_1} \right|^{-1/2} e^{-i3\pi/4} (M\delta)^{1/2} \left[\frac{u_0'(T_0+c^2)}{c(T_0-c^2)} + \frac{T_0}{c^2} \left(\left(\frac{(Mk-\omega)^2}{\sqrt{k^2-(Mk-\omega)^2}} \right)^{-1} + \left(\int_0^\delta \frac{T_0 dz}{(u_0-c)^2} - \delta \right) \right)^{-1} \right] \right),$$

where all functions are taken at $z = 0$.

Substituting this into (2), we obtain the dispersion relation:

$$\mathcal{D}(k,\omega) = Dk^4 + M_w^2 k^2 - \omega^2 - \mu \left(\left(\frac{(Mk-\omega)^2}{\sqrt{k^2-(Mk-\omega)^2}} \right)^{-1} + \left(\int_0^\delta \frac{T_0 dz}{(u_0-c)^2} - \delta \right) \right)^{-1} \times \left(1 - \varepsilon \left| \frac{\rho c}{\mu_1} \right|^{-1/2} e^{-i3\pi/4} (M\delta)^{1/2} \left[\frac{u_0'(T_0+c^2)}{c(T_0-c^2)} + \frac{T_0}{c^2} \left(\left(\frac{(Mk-\omega)^2}{\sqrt{k^2-(Mk-\omega)^2}} \right)^{-1} + \left(\int_0^\delta \frac{T_0 dz}{(u_0-c)^2} - \delta \right) \right)^{-1} \right] \right) = 0 \quad (20)$$

Using Taylor's expansion in the small parameter μ, we obtain the solution of the dispersion relation

$$\omega(k,\mu) = \omega(k,0) + \mu \left. \frac{\partial \omega}{\partial \mu} \right|_{\mu=0} + o(\mu) = \omega(k,0) - \mu \left. \left(\frac{\partial \mathcal{D}}{\partial \mu} \Big/ \frac{\partial \mathcal{D}}{\partial \omega} \right) \right|_{\mu=0} + o(\mu).$$

Omitting $o(\mu)$, we find

$$\omega(k,\mu) = \omega(k,0) - \frac{\mu}{2\omega(k,0)} \left(\left(\frac{(Mk-\omega)^2}{\sqrt{k^2-(Mk-\omega)^2}} \right)^{-1} + \left(\int\limits_0^\delta \frac{T_0 dz}{(u_0-c)^2} - \delta \right) \right)^{-1}$$

$$\times \left[1 - \varepsilon \left(\frac{\rho c}{\mu_1} \right)^{-1/2} e^{-i3\pi/4} (M\delta)^{1/2} \left[\frac{u_0^{'}(T_0+c^2)}{c(T_0-c^2)} \right.\right.$$

$$\left.\left. + \frac{T_0}{c^2} \left(\left(\frac{(Mk-\omega)^2}{\sqrt{k^2-(Mk-\omega)^2}} \right)^{-1} + \left(\int\limits_0^\delta \frac{T_0 dz}{(u_0-c)^2} - \delta \right) \right)^{-1} \right] \right]. \quad (21)$$

As before, all the expressions are taken at $z = 0$, $\mu = 0$.

5.3 The effect of finite Reynolds numbers on growing waves

Consider waves satisfying the following conditions:

1. In the absence of the boundary layer (i.e., in uniform flow) the wave is growing, i.e. $0 < c < M - 1$
2. In the presence of the boundary layer the wave in case of $R = \infty$ grows faster than in the uniform flow (see Section 4.2)
3. Let $\operatorname{Re} B < 0$, which is a necessary condition (14) for the stability of the boundary layer over a rigid plate [9].
4. We assume that $(T_0(0) - c^2) > 0$ for $0 < c < M - 1$. It can be shown that for $Pr = 1$ and $\gamma < 3$ (this case holds for most gases) this condition is satisfied for $M < 4/(3-\gamma)$, which is $M < 2.5$ for $\gamma = 1.4$.

For such waves, representing the "worst" case (growing wave, for which the shear layer yields the increase of the growth rate), we figure out whether the viscosity yields additional growth or damping. Using (21) and the above-mentioned assumptions, we obtain:

$$\omega(k,\mu) = \omega(k,0) - \frac{\mu}{2\omega(k,0)} (A+B)^{-1} \left(1 - \varepsilon K_1 e^{-i3\pi/4} \left[K_2 + K_3 (A+B)^{-1}\right]\right) \Bigg|_{z=0,\ \mu=0},$$

where

$$K_1 = \left| \frac{\rho c}{\mu_1} \right|^{-\frac{1}{2}} (M\delta)^{\frac{1}{2}} > 0, \quad K_2 = \frac{u_0^{'}(T_0+c^2)}{c(T_0-c^2)} > 0, \quad K_3 = \frac{T_0}{c^2} > 0,$$

and A and B are the appropriately defined functions.

The influence of finite Reynolds number on $\operatorname{Im}\omega$ is expressed in the sign of the term of the order of ε, which equals to

$$\operatorname{sign}(\Delta_\varepsilon \operatorname{Im}\omega) = \operatorname{sign}\operatorname{Im}\left((A+B)^{-1} K_1 e^{-i3\pi/4} \left[K_2 + K_3 (A+B)^{-1}\right]\right) \Bigg|_{z=0,\ \mu=0}. \quad (22)$$

5.4 Results

Investigation of (22) shows that in the case of small boundary layer thickness, the viscosity yields additional increase of the growth rate. Thus, such waves are indeed "worst", because the viscosity has destabilizing effect comparing to both uniform flow and inviscid shear layer.

For more general case than considered in the previous section, it can be shown that the viscosity can yield both stabilizing and destabilizing effect, which depends on the wave number and the boundary layer flow profile.

6 CONCLUSIONS

In this paper, we derived a dispersion relation for a plate in supersonic flow with the boundary layer over the plate taken into account, for both cases of inviscid perturbations of the boundary layer and for viscous perturbations at large Reynolds numbers.

In the inviscid case, for "moderate" wave lengths an analytical solution $\omega(k)$ is derived, and the influence of the boundary layer on the plate stability is analysed. For short waves, when the Rayleigh equation cannot be simplified and solved analytically, numerical study is conducted. It is shown that the term of the order of k^2 in the Rayleigh equation yields additional destabilisation for a certain segment of the boundary layer thicknesses.

For the case of large finite Reynolds numbers, the analysis of the growth rate is conducted for the "worst" waves, which are growing in uniform flow, and the boundary layer at infinite Reynolds number has additional destabilizing effect. It was found that in the case of a small boundary layer thickness, finite Reynolds number yields even more destabilization. In the general case, finite Reynolds number can have both stabilizing and destabilizing effect, depending on the wave length and the flow parameters.

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