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Influence of longitudinal tension on the stability of the finite length elastic tubes conveying Non-Newtonian fluid

Influence of longitudinal tension on the stability of the finite length elastic tubes conveying Non-Newtonian fluid

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Abstract. In theoretical and experimental studies of flow instabilities in collapsible tubes, the loss of stability occurs in the form of non-axisymmetric motion of the tube walls with the partial or full collapse of the tube, while the axisymmetric perturbations of the tube are damped. However, for the case of non-Newtonian power law fluid, axisymmetric perturbations can grow for a small power law index *n*. An analysis of infinite-length tube shows that instability is possible only for power law index $n<0.611$, and absolute instability can occur only for $n<1/3$. This paper is devoted to investigation of flutter of finite length elastic tubes conveying power law fluid. For untensioned finite length tubes an analytical solution is obtained. The instability boundary coincides with the boundary of absolute instability for infinitely long tubes. For non-zero longitudinal tension *N* the problem is investigated numerically. It is shown that the region of instability becomes smaller with an increase of the longitudinal tension *N*.

1. Introduction

At the present day, the modelling of elastic tubes conveying Newtonian fluids is used for studying the dynamics of biological fluids. Oscillations of tube walls are responsible for flow and pressure drop limitations that play an important role in biology [1, 2]. That is why flutter of elastic tubes has been of great interest over several decades [3-5]. It is known, however, that blood in small vessels [6], and pathological gall [7] can show essentially Non-newtonian properties.

The simplest theoretical one-dimensional models were first used to analyse the stability of elastic tubes in [8, 9]. Each tube is characterized by the correspondence of the cross-sectional area to the transmural pressure (the difference between internal and external pressures) known as the tube law. When the transmural pressure is negative and the flow velocity is large, the tube loses stability and collapses. During each tube collapse, the deformation of the tube is essentially non-axisymmetric, and the flow inside of the tube is rather complex.

This paper is devoted to reinvestigation of flutter of elastic tubes conveying Non-Newtonian fluid based on a simple one-dimensional model. The nature of axisymmetric instability is studied.

The stability problem for finite length tube without taking the longitudinal tension into account is investigated analytically. For non-zero longitudinal tension the problem is studied numerically.

2. Formulation of the problem

2.1. Equation of motion

Consider a cylindrical tube of length L with elastic walls fixed on rigid cylindrical tubes of lengths L_1, L_2 and non-Newtonian fluid flowing inside (figure 1). The flow and the tube interact with each other through to the equality of normal stress at the wall and matching of the flow region with the shape of the tube. The longitudinal motion of the tube wall is neglected. The flow is assumed to be axisymmetric.

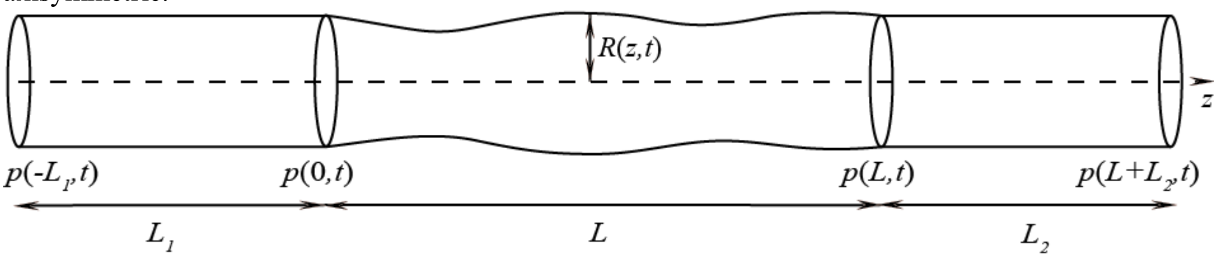

Figure 1. Axisymmetric perturbation of the elastic tube.

Introduce cylindrical coordinate system with z axis directed along the tube. We assume that the tube motion is so that each cross-section $S(z,t)$ remains circular, and the tube points move only in the radial direction. Then the tube geometry is defined by its radius $R(z,t)$ at the point z and time t.
System of equations for the incompressible fluid motion free from body forces is written as:

$$\begin{cases} div\left(\vec{v}\right)=0; \\ \dfrac{dv^i}{dt}=-\dfrac{1}{\rho}\dfrac{\partial p}{\partial x^i}+\dfrac{1}{\rho}\nabla_j\tau^{ij}, \quad i=1,2,3. \end{cases} \tag{1}$$

where ρ is the fluid density, v is the fluid velocity, x^i is coordinate axes.

Taking tube inertia and longitudinal tension of the tube wall into account, the motion of the tube wall is written as:

$$\beta(R-R_0)-N\frac{\partial^2 R}{\partial z^2}=P-m\frac{\partial^2 R}{\partial t^2}, \tag{2}$$

where $\beta=Eh\big/\left[(1-\upsilon^2)R_0^2\right]$ is the radial stiffness of the tube, m is wall surface density, $N=\sigma h$ is longitudinal tension, R_0 is undeformed tube radius, E and υ are Young's modulus and Poisson coefficient, h is the tube wall thickness, σ is the tube tension stress, P is flow pressure.

The fluid rheology obeys power law:

$$\tau^{ij}=2^{(n+1)/2}\mu\left(I_2\left(e\right)\right)^{n-1}e^{ij},\ I_2=\left(e^{ij}e_{ij}\right)^{1/2}. \tag{3}$$

Where τ^{ij} is the viscous stress tensor, e^{ij} is the strain velocity tensor.

The law (3) is a generalization to a three-dimensional case of the pure shear relationship:

$$\tau^{12}=\mu\left(\frac{dv_1}{dv_2}\right)^n.$$

2.2. One-dimensional system of equation.

We will consider perturbations satisfying two conditions:

- wave length λ is large comparing to the tube radius R;

- wave frequency is sufficiently small so that the flow inside the tube can be considered as quasi-steady, so that a Poiseuille velocity profile [10] is formed in each cross-section

$$v_z(r) = \frac{Q}{\pi R^2}\frac{1+3n}{1+n}\left(1-(r/R)^{(n+1)/n}\right) \tag{4}$$

In the case of Newtonian fluid ($n=1$) we obtain a standard parabolic flow.

The system (1), (2), (3) is reduced to one-dimensional problem by integration over the tube cross-section:

$$\begin{aligned}
&\frac{\partial Q}{\partial z}+\frac{\partial \pi R^2}{\partial t}=0,\\
&\frac{\partial Q}{\partial t}+\frac{\partial}{\partial z}\left(\frac{(3n+1)Q^2}{(2n+1)\pi R^2}\right)+\frac{\mu}{\rho}\frac{2(3n+1)^n Q^n}{n^n\pi^{n-1}R^{3n-1}}+\frac{\pi R^2}{\rho}\frac{\partial p}{\partial z}=0,\\
&p=\beta(R-R_0)-N\frac{\partial^2 R}{\partial z^2}+m\frac{\partial^2 R}{\partial t^2}.
\end{aligned} \tag{5}$$

2.3. *Dimensionless problem*

Let us choose the fluid density ρ, tube radius R_0, and flow rate Q_0 at the tube inlet as independent scales. The dimensionless parameters (denoted by a tilde) are expressed in terms of the dimensional parameters:

$$\tilde{\beta}=R_0^5\beta\big/\left(\rho Q_0^2\right),\ \tilde{N}=R_0^3N\big/\left(\rho Q_0^2\right),\tilde{P}_{in}=R_0^4P_{in}\big/\left(\rho Q_0^2\right),\ \tilde{\mathrm{Re}}=\rho R_0^n 8n^n\left(Q_0\big/\left(\pi R_0^2\right)\right)^{2-n}\Big/\left(\mu(3n+1)^n\right),$$

where $\tilde{\mathrm{Re}}$ is the generalized Reynolds number of Metzner-Reed [11].

Substitute *P* expressed through *R* (5) and omitting tildes, we obtain the dimensionless system of two one-dimensional equations:

$$\begin{aligned}
&\frac{\partial Q}{\partial z}+\frac{\partial \pi R^2}{\partial t}=0,\\
&\frac{\partial Q}{\partial t}+\frac{\partial}{\partial z}\left(\frac{(3n+1)Q^2}{(2n+1)\pi R^2}\right)+\frac{16Q^n}{\pi\,\mathrm{Re}\,R^{3n-1}}+\pi R^2\beta\frac{\partial R}{\partial z}+\pi R^2 m\frac{\partial^3 R}{\partial z\partial t^2}-\pi R^2 N\frac{\partial^3 R}{\partial z^3}=0.
\end{aligned} \tag{6}$$

3. Method of analysis

For the wave length λ we will assume that $\lambda \ll L$, where is the tube length *L*. We consider travelling wave solutions so that

$$Q=1+Q'e^{i(kz-\omega t)}$$
$$R=1+R'e^{i(kz-\omega t)}$$

Substituting into (6) and linearizing, we obtain the dispersion relation

$$\left(1+\frac{mk^2}{2}\right)\omega^2+\left(\frac{16n}{\pi\mathrm{Re}}i-\frac{2(3n+1)k}{\pi(2n+1)}\right)\omega+\frac{8(1-3n)k}{\pi^2\,\mathrm{Re}}i-\frac{k^2\beta}{2}+\frac{k^2(3n+1)}{\pi^2(2n+1)}-\frac{Nk^4}{2}=0. \tag{7}$$

The instability criterion is that there exists a real wave number *k*, such that the imaginary part of frequency $\mathrm{Im}\,\omega(k)$ is positive.

4. Local instability of an infinite length tube

Analysis of the dispersion relation yields the following instability criterion:

$$\beta < \frac{-(6n^3 - n^2 - 1)}{2n^2\pi^2(2n+1)}. \tag{8}$$

Since $\beta > 0$, instability can occur only if the right-hand side is positive. This is the case for $n < 0.611$ (figure 2).

Even if the tube conveying pseudoplastic fluid is unstable, this instability might not be observed if the instability is convective. In this case localized perturbations grow but also travel along the tube and can leave it while having small amplitude. However, if the instability is absolute, the wave grows around the initial disturbance and eventually spread out to every point in space. Analysis shows that the criterion of absolute instability is

$$\frac{2(3n+1)}{\pi^2(2n+1)} < \beta < \frac{2(3n+1)(1-n)}{\pi^2 n(2n+1)}. \tag{9}$$

This condition can be satisfied only for $n < 1/3$ (figure 3).

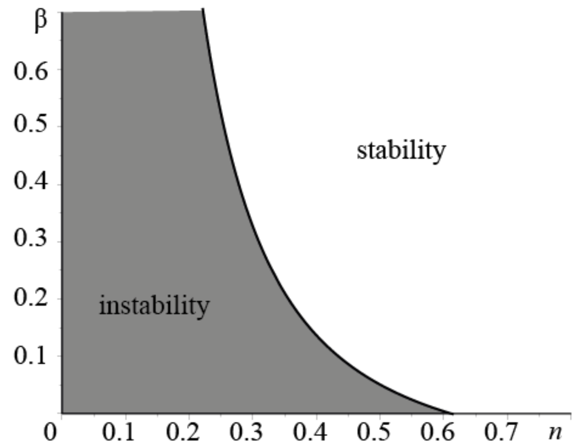

Figure 2. Instability region.

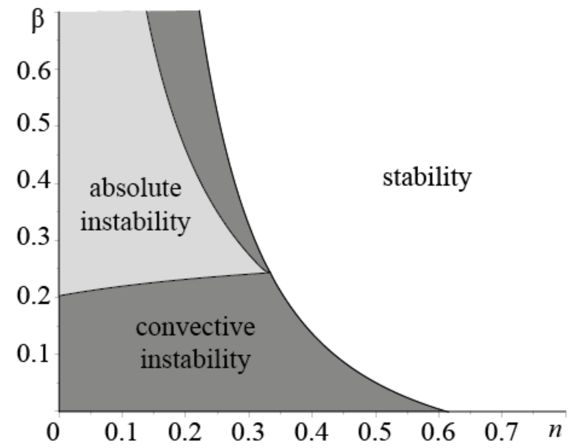

Figure 3. Regions of absolute and convective instability.

5. Instability of a finite length tube

The dispersion equation (7) for the wave number *k* is a fourth-order equation, so it is necessary to specify four boundary conditions. Since the elastic tube is fixed at the input and output section, we obtain the first two boundary conditions from the first equation of the system (6):

$$\frac{\partial Q(0)}{\partial z} = \frac{\partial Q(L)}{\partial z} = 0. \tag{10}$$

In addition, the pressure is set on inlet and outlet of rigid sections of the tube (figure 1); which is transferred to inlet and outlet of the collapsible segment as:

$$p(0,t) = p(-L_1,t) - L_1\left(\frac{\partial Q}{\partial t} + \frac{16Q^n}{\pi \operatorname{Re} R^{3n-1}}\right)\frac{1}{\pi R^2};$$
$$p(L,t) = p(L+L_2,t) + L_2\left(\frac{\partial Q}{\partial t} + \frac{16Q^n}{\pi \operatorname{Re} R^{3n-1}}\right)\frac{1}{\pi R^2}. \tag{11}$$

We consider eigenmode solution: $Q = e^{-i\omega t}\sum_{j=1}^{4} Q_j e^{ik_j z}$, where k_j are different roots of the dispersion equation and Q_j are constant.

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According to the method of analysis similar to [12], to satisfy the boundary conditions, it is necessary that $\det M = 0$, where the matrix M is:

$$M_{1j} = ik_j;$$
$$M_{2j} = ik_j e^{ik_j L};$$
$$M_{3j} = -\frac{N}{2\pi} ik_j^3 - \frac{L_1}{\pi}\left(\omega^2 + \frac{16ni}{\pi \mathrm{Re}}\omega\right);$$
$$M_{4j} = -\frac{N}{2\pi} ik_j^3 e^{ik_j L} + \frac{L_2}{\pi}\left(\omega^2 + \frac{16ni}{\pi \mathrm{Re}}\omega\right)e^{ik_j L}.$$

To find the region of instability the problem is solved numerically. However, without taking into account the longitudinal tension and the surface density of the tube (i.e., for $N = 0, m = 0$), the problem is solved analytically.

6. Results

Without taking the longitudinal tension N and wall surface density m into account, the instability boundary for the tube of finite length coincides with the absolute instability boundary for infinitely long tubes. When the longitudinal tension is taken into account, the instability region decreases with increasing value of the longitudinal tension N (figure 4).

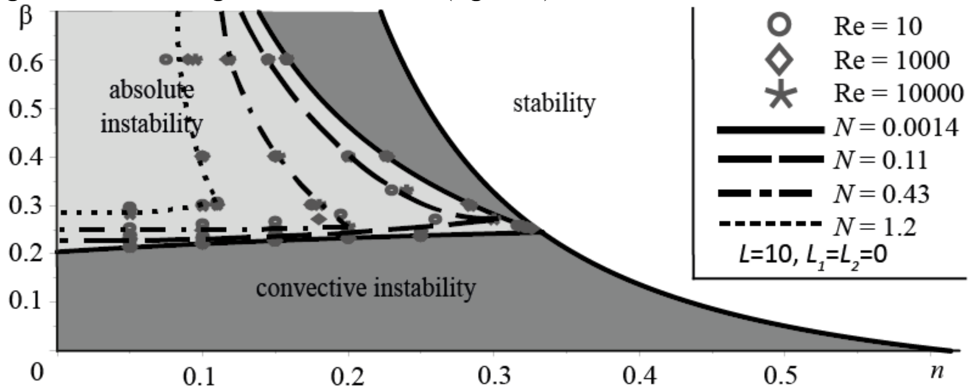

Figure 4 Instability region for L=10.

On the other hand, the increase of the elastic tube length has a destabilizing effect, i.e., the instability region is larger for longer tubes under the same longitudinal tension N (figure 5).

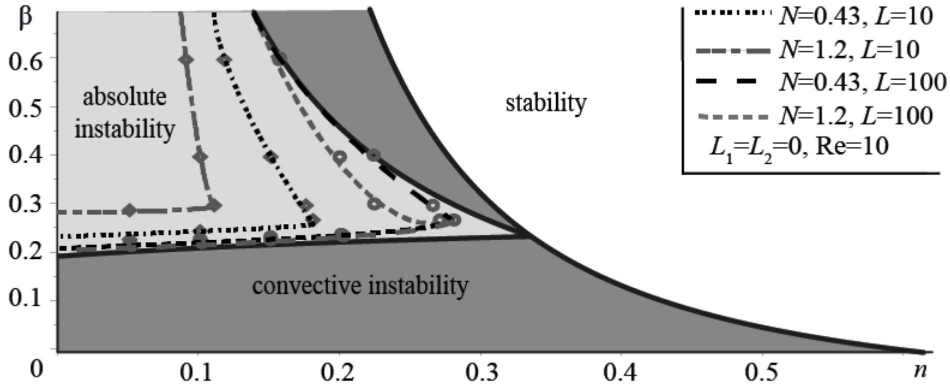

Figure 5. Instability region for different L.

However, the stabilizing effect occurs when the lengths of the rigid input and output tubes increase, i.e. under the same values of the longitudinal tension *N* the region of instability is greater for smaller the lengths of the rigid tubes (figure 6).

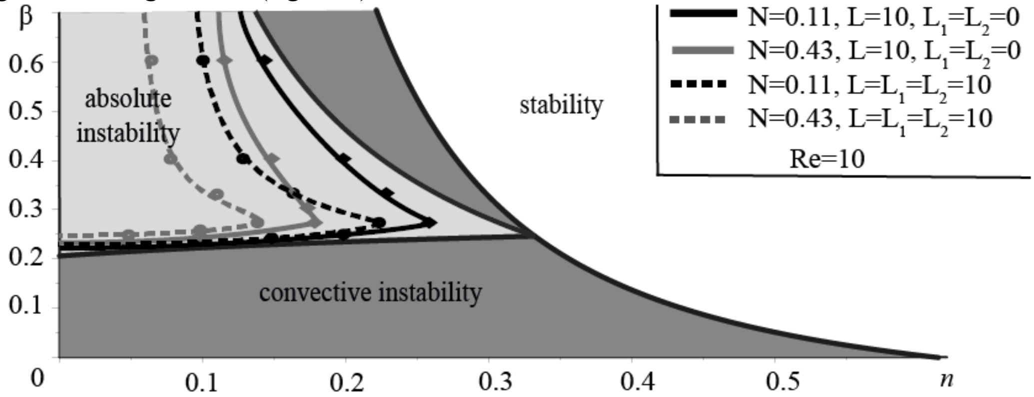

Figure 6. Instability region for different L_1, L_2 .

7. Conclusions

The system of two one-dimensional equations is obtained for the local flow rate Q and the radius *R* for axisymmetric perturbations. Local instability criterion for axisymmetric perturbations of power law fluid flows in elastic tubes is obtained. Instability is possible only for $n < 0.611$. Absolute instability can occur only for $n < 1/3$.

It is shown that for a tube of finite length without taking the longitudinal tension *N* into account, the instability boundary coincides with the absolute instability boundary for infinitely long tubes. When the longitudinal tension *N* is taken into account, the instability region decreases with increasing longitudinal tension *N*.

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