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On limits of applicability of the homogenization method to modeling of layered creep media

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Abstract: The paper considers the homogenized model of the layered creep media. It is shown that the homogenized model is sufficiently accurate. The difference is less than 5% as compared with the results obtained with the use of direct computations when non-homogenized model is utilized. This tolerance holds for rather long observation time. The medium sample can be deformed by 10-15% of its size by the end of the considered time period. It is shown that 10 layers of the medium are sufficient for obtaining the described accuracy of the homogenized model. Therefore, application of the homogenized model allows to decrease the number of computation elements by 4 orders.

Keywords: modeling in hereditary continuum mechanics, homogenization, creep, composite media, layered media modeling, time steps method

1. INTRODUCTION

A large number of composite material models was considered in the theory of homogenization. The composite material is given by its periodic structure and by physical characteristics of its components. The goal of the homogenization is to derive average properties of the composite medium from the properties of these components. When modeling the medium, this approach allows to significantly decrease the computation time, because it allows to use a sparse computational mesh for the whole homogenized medium instead of a fine mesh on each medium component. Classical works on the topic (Sanchez-Palencia (1980), Bakhvalov and Panasenko (1989), Oleinik et al. (1992)) consider the models described by the partial differential equations. More recent studies also adress the models with integro-differential equations. Homogenization for a creep media was examined by Orlik (2000). The procedure of homogenization was implemented here for the viscoelastic medium described by integro-differential relations. The expressions for the homogenized medium properties were obtained for a periodic in three dimensions media, but this expressions were rather complex and they needed for considerable computation resources, because they required solving auxiliary problems on the cell of periodicity. Another approach to homogenization of a creep composite was proposed by Shamaev and Shumilova (2016) for a layered medium. Here, relatively simple expressions were derived for determining the homogenized medium properties. These expressions only assumed solution of one-dimensional integral equations.

The present paper implements the homogenization approach proposed by Shamaev and Shumilova (2016) for the layered creep media. Cubic sample deformations are examined for different load types. Simulations are performed both for the layred media directly and for the homogenized model. The results are compared. Our goal is to determine the applicabilty of the homogenized model for the creep composite modeling for the sufficiently long observation times. In both cases (direct computations and the homogenized model), the simulation is performed with the use of the time steps method proposed by Konstantinova and Chernopazov (2004, 2006, 2007).

2. PROBLEM STATEMENT

The considered layered media consists of two materials. Each of these materials obeys the rheological law of hereditary creep, see Konstantinova and Chernopazov (2004):

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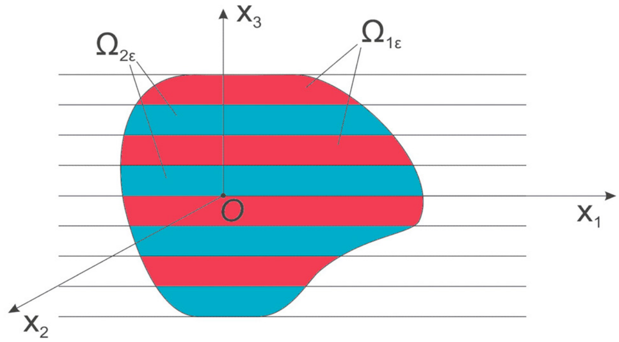

Fig. 1. Scheme of the layered medium

$$s_{ij}(t) = s_{ij}(0) + 2Ge_{ij}(t) - \int_0^t K_C(\widetilde{\sigma}(\tau),\, t-\tau)s_{ij}(\tau)d\tau;$$
$$p(t) = p(0) + B\theta(t) - \int_0^t K_V(\widetilde{\sigma}(\tau),\, t-\tau)p(\tau)d\tau, \tag{1}$$

where $p = \frac{1}{3}\sigma_i^i$ is the pressure, $\theta = \varepsilon_i^i$ is the volume deformation, s_{ij} and e_{ij} are the components of deviators of the stress tensor σ_{ij} and of the strain tensor ε_{ij} defined by

$$s_{ij} = \sigma_{ij} - p\delta_{ij},$$
$$e_{ij} = \varepsilon_{ij} - \frac{1}{3}\theta\delta_{ij},$$

where δ_{ij} is the Kronecker delta, $B = \dfrac{E}{3(1-2\nu)}$ is the bulk modulus, $G = \dfrac{E}{2(1+\nu)}$ is the shear modulus (E and ν are the Young's modulus and Poisson's ratio, which describe the elastic properties of the material), K_C and K_V are the shear and bulk creep kernels, $\widetilde{\sigma}(\tau)$ is the stress tensor at the time instant τ. The creep kernels are considered to be the Abel kernels:

$$K_i(\widetilde{\sigma}(\tau),\, t-\tau) = \delta_i(\widetilde{\sigma}(\tau))(t-\tau)^\alpha, \quad i = 1, 2, \tag{2}$$

where α is the material constant, $\delta_i(\cdot)$ is some function. The subscript 1 denotes the first creep material, and the subscript 2 denotes the second one. Values $-1 < \alpha < 0$ correspond to physically possible cases. For simplicity, we assume

$$\delta_C \equiv \text{const},$$
$$\delta_V \equiv \text{const}.$$

For further considerations, the stress-strain relations (1) should be reversed. As a result of this inversion, the media strains will be the integrand instead of the media stresses. Let us rewrite the stress-strain relation (1) as follows

$$e_{ij} = \frac{1}{2\mu}(s_{ij} + K_c(t) * s_{ij}),$$
$$\theta = \frac{1}{K}(p + K_v(t) * p),$$

where $K_c(t) = \delta_c t^\alpha$, $K_v(t) = \delta_v t^\alpha$ are the kernels of shear and bulk creep, the symbol $*$ stands for the convolution of two functions, $K \equiv B$, $\mu \equiv G$. Then, the inversion is given by the following expression for each material (the index of the material is omitted here):

$$\sigma_{ij} = a_{ijkh}\varepsilon^{kh} + d_{ijkh}(t) * \varepsilon^{kh}, \tag{3}$$

$$a_{ijkh} = \lambda\delta_{ij}\delta_{kh} + \mu(\delta_{ik}\delta_{jh} + \delta_{ih}\delta_{jk}), \tag{4}$$

$$d_{ijkh}(t) = -\left(k_g - \frac{1}{3}\right)G_c(t)\delta_{ij}\delta_{kh} - \frac{1}{2}G_c(t)(\delta_{ik}\delta_{jh} + \delta_{ih}\delta_{jk}). \tag{5}$$

The kernels of shear and bulk relaxation have the form

$$G_c(t) = C_0 \mathrm{R}_\alpha(\beta, t),$$
$$G_v(t) = k_g G_c(t),$$

where $R_\alpha(\beta, t)$ is the Rabotnov function, see Rabotnov (1969). This function is expressed through the generalized Mittag-Leffler function as in Gorenflo et al. (2014):

$$\mathrm{R}_\alpha(\beta, t) = t^\alpha E_{\alpha+1,\alpha+1}(\beta t^{\alpha+1}),$$

where the Mittag-Leffler function

$$E_{\alpha_1,\alpha_2}(t) = \sum_{n=0}^{\infty} \frac{t^n}{\Gamma(\alpha_1 n + \alpha_2)}, \tag{6}$$

and $\Gamma(t)$ is the Eulerian Gamma function. The parameters C_0, k_g, β are defined below.

3. COMPUTATIONAL ALGORITHM

The method of time steps (Konstantinova and Chernopazov (2004, 2006, 2007)) is used for determining the strains as functions of time under the given stresses applied to the sample. The homogenized stress-strain relation is needed on the each computational step, so, the method of homogenization proposed by Shamaev and Shumilova (2016) is used to determine the average strain tensor and the average creep kernels of the layered medium.

3.1 The method of time steps in the problem of layered creep composites homogenization

Let the creep kernels be linear in stresses or be independent from them. The original stress-strain relation can be represented in the more general tensor form

$$\varepsilon_{ij}(t) = b_{ijhl}\sigma_{hl} + \int_0^t d_{ijhl}(t-\tau)\sigma_{hl}(\tau)d\tau.$$

The constitutive equation is written in the form

$$\sigma_{ij}(t_k) = a_{ijhl}^{(k,k-1)}\varepsilon_{hl}(t_k) - \int_0^{t_{k-1}} g_{ijhl}^{(k,k-1)}(t-\tau)\sigma_{hl}(\tau)d\tau \tag{7}$$

with the use of the approximate equality

$$\int_{t_{k-1}}^{t_k} d_{ijhl}(t_k-\tau)\sigma_{hl}(\tau)d\tau \cong \widetilde{d}_{ijhl}^{(k,k-1)}\sigma_{hl}(t_k),$$

where the value of $\widetilde{d}_{ijkl}$ is known from the problem statement.

According to the method of time steps from Konstantinova and Chernopazov (2004), we assume

$$\Xi_{ij}^{(k-1)} = \int_0^{t_{k-1}} g_{ijhl}^{(k,k-1)}(t_k, \tau)\sigma_{hl}(\tau)d\tau. \tag{8}$$

The equilibrium equations at time $t = t_k$ take the form

$$\frac{\partial \sigma_{ij}}{\partial x_i}(x, t_k) \equiv \frac{\partial}{\partial x_i}(a_{ijhl}^{(k,k-1)} \varepsilon_{hl}(t_k) - \Xi_{ij}^{(k-1)}) = f_j(x), \quad (9)$$

where $f_j(x)$ are the components of the body forces, $\Xi_{ij}^{(k-1)}$ is the known function that is obtained from the values calculated at the previous time steps.

3.2 Computation of the homogenized stress-strain relations

The homogenized medium characteristics are defined as follows, see Shamaev and Shumilova (2016). First, for each material we calculate the following values:

- $\lambda_i = \frac{E_i \nu_i}{(1+\nu_i)(1-2\nu_i)}$ is the first Lame parameter, Pa,
- $\mu_i = \frac{E_i}{2(1+\nu_i)}$ is the second Lame parameter (the shear modulus), Pa,
- $K_i = \lambda_i + \frac{2}{3}\mu_i$ is the module of full compression (the bulk modulus of elasticity), Pa,
- $C_{0,i} = 2\mu_i \delta_{c,i} \Gamma(1+\alpha)$ are the constant multipliers in the kernels of the share relaxation, $\mathrm{Pa} \cdot \mathrm{h}^{-\alpha-1}$.
- $k_{g,i} = \frac{K_i \delta_{v,i}}{2\mu_i \delta_{c,i}}$ are the dimensionless constants of proportionality in the kernels of the share and bulk relaxation,
- $\beta_i = -\delta_{c,i}\Gamma(1+\alpha)$ are the parameters of the Rabotnov function, $\mathrm{h}^{-\alpha-1}$,

and the following auxiliary constants:

- $a_1 = \lambda_1 + 2\mu_1, \quad a_2 = \lambda_2 + 2\mu_2$,
- $A = a_1 H + a_2(1-H), \quad B = \mu_1 H + \mu_2(1-H)$,
- $c_{11} = \frac{(a_2-a_1)(1-H)}{A}$, $c_{22} = \frac{(\lambda_2-\lambda_1)(1-H)}{A}$, $c_{12} = \frac{(\mu_2-\mu_1)(1-H)}{B}$.

Here again, the subscripts 1 and 2 correspond to the first and second materials respectively.

Then after reversing the given rheology relations, they are written in the tensor form:

$$\sigma_{ij}(x,t) = a_{ijkh}(x)\varepsilon_{kh}(x,t) + d_{ijkh}(x,t) * \varepsilon_{kh}(x,t),$$

The elastic tensor a_{ijkh}, the strain tensor $\varepsilon_{kh}(x,t)$ and the creep tensor d_{ijkh} now depend on the spatial coordinate x, because the medium has layers. Depending on the value of coordinate x, the properties of the first or second material are used. As a result, all tensors oscillate as periodic piecewise constant functions.

Here is the function that defines the elastic oscillating tensor:

$$a_{ijkh}(x) = \lambda(x)\delta_{ij}\delta_{kh} + \mu(x)(\delta_{ik}\delta_{jh} + \delta_{ih}\delta_{jk}).$$

The function that defines the creep oscillating tensor, is expressed as:

$$\begin{aligned} d_{ijkh}(x,t) = &-\left(k_g(x) - \frac{1}{3}\right) G_c(x,t)\delta_{ij}\delta_{kh} \\ &-\frac{1}{2}G_c(x,t)(\delta_{ik}\delta_{jh} + \delta_{ih}\delta_{jk}). \end{aligned} \quad (10)$$

In contrast to the expressions (3), (4), and (5), now σ_{ij}, a_{ijkh}, and d_{ijkh} depend on the spatial coordinate x.

We need three auxiliary functions $p_{ij}(t)$ for calculating the averaged elasticity and creep tensors. In order to find these functions, the following integral equations should be solved (the indexes (1) and (2) denote the tensor values in the first and second media correspondingly):

$$\begin{aligned} &p_{11}(t) + \frac{1}{A}\left(Hd_{1111}^{(1)}(t) + (1-H)d_{1111}^{(2)}(t)\right) * p_{11}(t) \\ &= \frac{1-H}{A^2}\left(-a_2 d_{1111}^{(1)}(t) + a_1 d_{1111}^{(2)}(t)\right), \end{aligned} \quad (11)$$

$$\begin{aligned} &p_{22}(t) + \frac{1}{A}\left(Hd_{1111}^{(1)}(t) + (1-H)d_{1111}^{(2)}(t)\right) * p_{22}(t) \\ &= -\frac{1}{A}\Big(c_{22}(Hd_{1111}^{(1)}(t) + (1-H)d_{1111}^{(2)}(t)) \\ &\quad +(1-H)(d_{1122}^{(1)}(t) - d_{1122}^{(2)}(t))\Big), \end{aligned} \quad (12)$$

$$\begin{aligned} &p_{12}(t) + \frac{1}{B}\left(Hd_{1212}^{(1)}(t) + (1-H)d_{1212}^{(2)}(t)\right) * p_{12}(t) \\ &= \frac{1-H}{B^2}\left(-\mu_2 d_{1212}^{(1)}(t) + \mu_1 d_{1212}^{(2)}(t)\right). \end{aligned} \quad (13)$$

Thus, it is necessary to solve three integral equations of the type

$$f(t) + \int_0^t K(t-s)f(s)ds = g(t), \quad t \in [0,b], \quad (14)$$

where the functions $K(t)$ and $g(t)$ are given, and $f(t)$ should be found.

For the numerical solution of (14), the interval $[0,T]$ is divided into N subintervals that have the lengths $h = \frac{T}{N}$. These subintervals have ends in points $x_i = \frac{T}{N}i$, where $i = 0, ..., N$.

Functions f and g are given as the vector-columns

$$f^N = \{f(x_0), f(x_1), ..., f(x_N)\}^T,$$

$$g^N = \{g(x_0), g(x_1), ..., g(x_N)\}^T.$$

The trapezoid method is used for the numerical approximation of the integral in (14) on intervals $[x_i, x_{i+1}]$. Then, the original equation (14) takes the form of the system of linear equations

$$Af^N = g^N, \quad (15)$$

where the matrix A is expressed according to

$$A = \begin{pmatrix} 1 & 0 & 0\ 0\ . & 0 \\ \frac{hK(x_1-x_0)}{2} & \frac{hK(0)}{2}+1 & 0\ 0\ . & 0 \\ . & . & .\ .\ . & . \\ \frac{hK(x_i-x_0)}{2} & hK(x_i-x_1) & .\ .\ . & 0 \\ . & . & .\ .\ . & . \\ \frac{hK(x_N-x_0)}{2} & hK(x_N-x_1) & .\ .\ . & \frac{hK(0)}{2}+1 \end{pmatrix}.$$

It is evident that for the regular convolution kernels $K(t)$ and for the sufficiently small step h, the matrix A is close to the diagonal one, with all the diagonal elements equal to 1. For kernels with a singularity of the form t^α, $\alpha < 0$, the integration should be slightly different at the first step (since $K(0) = \infty$).

Let us consider the auxiliary elastic tensors, see Shamaev and Shumilova (2016):

$$\varepsilon_{ij}(Z^{kh}) = \frac{1}{2}\left(\frac{\partial Z_i^{kh}}{\partial x_j} + \frac{\partial Z_j^{kh}}{\partial x_i}\right),$$

where

$$\begin{aligned} &Z^{11}(y) = (c_{11}Z(y), 0, 0),\\ &Z^{22}(y) = Z^{33}(y) = (c_{22}Z(y), 0, 0),\\ &Z^{12}(y) = Z^{21}(y) = (0, c_{12}Z(y), 0),\\ &Z^{13}(y) = Z^{31}(y) = (0, 0, c_{12}Z(y)),\\ &Z^{23}(y) = Z^{32}(y) = (0, 0, 0), \end{aligned}$$

$y \equiv x_1$. Here the function

$$Z(y) = \begin{cases} \dfrac{Hy}{1-H}, & 0 \leqslant y < (1-H)/2,\\ \dfrac{1}{2} - y, & (1-H)/2 \leqslant y < (1+H)/2,\\ \dfrac{(y-1)H}{1-H}, & (1+H)/2 \leqslant y \leqslant 1; \end{cases}$$

so that

$$Z' = \begin{cases} \dfrac{H}{1-H}, & \text{in the material 1,}\\ -1, & \text{in the material 2.} \end{cases}$$

If $i \neq 1$ and $j \neq 1$, then $\varepsilon_{ij}(Z^{kh}) = 0$. Thus, the non-zero components are $\varepsilon_{11}(Z^{kh})$, $\varepsilon_{12}(Z^{kh})$, $\varepsilon_{13}(Z^{kh})$, and the components obtained from them by the i, j indexes permutations. The latter holds true due to the symmetry by these indexes. Substituting into the formulae, we get the expressions for these components:

$$\varepsilon_{11}(Z^{kh}) = \begin{pmatrix} c_{11}Z' & 0 & 0\\ 0 & c_{22}Z' & 0\\ 0 & 0 & c_{22}Z' \end{pmatrix},$$

$$\varepsilon_{12}(Z^{kh}) = \varepsilon_{21}(Z^{kh}) = \begin{pmatrix} 0 & c_{12}Z'/2 & 0\\ c_{12}Z'/2 & 0 & 0\\ 0 & 0 & 0 \end{pmatrix},$$

$$\varepsilon_{13}(Z^{kh}) = \varepsilon_{31}(Z^{kh}) = \begin{pmatrix} 0 & 0 & c_{12}Z'/2\\ 0 & 0 & 0\\ c_{12}Z'/2 & 0 & 0 \end{pmatrix}.$$

Likewise, the creep tensor components $\varepsilon_{ij}(W^{kh})$ are equal to zero if $i \neq 1$ and $j \neq 1$. The other components are equal to

$$\varepsilon_{11}(W^{kh}) = \begin{pmatrix} p_{11}(t)Z' & 0 & 0\\ 0 & p_{22}(t)Z' & 0\\ 0 & 0 & p_{22}(t)Z' \end{pmatrix},$$

$$\varepsilon_{12}(W^{kh}) = \varepsilon_{21}(W^{kh}) = \begin{pmatrix} 0 & p_{12}(t)Z'/2 & 0\\ p_{12}(t)Z'/2 & 0 & 0\\ 0 & 0 & 0 \end{pmatrix},$$

$$\varepsilon_{13}(W^{kh}) = \varepsilon_{31}(W^{kh}) = \begin{pmatrix} 0 & 0 & p_{12}(t)Z'/2\\ 0 & 0 & 0\\ p_{12}(t)Z'/2 & 0 & 0 \end{pmatrix}.$$

Finally, the homogenized stress-strain relations have the following form (the sign $\hat{}$ denotes an averaged value):

$$\begin{aligned} \hat{\sigma}_{ij}(x_1, x_2, x_3, t) &= \hat{A}_{ijkh}\hat{\varepsilon}_{kh}(x_1, x_2, x_3, t)\\ &+ \hat{B}_{ijkh}(t) * \hat{\varepsilon}_{kh}(x_1, x_2, x_3, t), \end{aligned} \tag{16}$$

where $\hat{A}_{ijkh}$ and $\hat{B}_{ijkh}(t)$ are the averaged values of the tensors

$$A_{ijkh}(x_1) = a_{ijkh}(x_1) + a_{ijlm}(x_1)\varepsilon_{lm}(Z^{kh}(y)),$$

$$\begin{aligned} B_{ijkh}(x_1, t) &= d_{ijkh}(x_1, t) + d_{ijlm}(x_1, t)\varepsilon_{lm}(Z^{kh}(y))\\ &+ a_{ijlm}(x_1)\varepsilon_{lm}(W^{kh}(y, t)) + d_{ijlm}(x_1, t) * \varepsilon_{lm}(W^{kh}(y, t)) \end{aligned}$$

calculated according to the formula

$$\hat{F} = (1-H)F_1 + HF_2$$

(the indexes 1 and 2 denote the values of F in materials 1 and 2 correspondingly).

The oscillating stress values are reconstructed from the known mean strains $\hat{\varepsilon}_{kh}(x_1, x_2, x_3, t)$:

$$\begin{aligned} \sigma_{ij}(x_1, x_2, x_3, t) &= A_{ijkh}(x_1)\hat{\varepsilon}_{kh}(x_1, x_2, x_3, t)\\ &+ B_{ijkh}(x_1, t) * \hat{\varepsilon}_{kh}(x_1, x_2, x_3, t). \end{aligned} \tag{17}$$

Note that the tensors A_{ijkh} and B_{ijkh} satisfy the symmetry relations

$$A_{ijkh} = A_{jikh} = A_{khij}, \quad B_{ijkh} = B_{jikh} = B_{khij}.$$

Moreover, only the following (and the ones obtained from their symmetry) components are non-zero:

$$\begin{gathered} A_{1111}, \quad A_{2222} = A_{3333}, \quad A_{1122} = A_{1133},\\ A_{2233}, \quad A_{1212} = A_{1313}, \quad A_{2323}, \end{gathered}$$

with

$$A_{2222} = 2A_{2233} + A_{2323}.$$

Similarly, only the following (and the ones obtained from their symmetry) creep tensor components are non-zero:

$$\begin{gathered} B_{1111}, \quad B_{2222} = B_{3333}, \quad B_{1122} = B_{1133},\\ B_{2233}, \quad B_{1212} = B_{1313}, \quad B_{2323},\\ B_{2222} = 2B_{2233} + B_{2323}. \end{gathered}$$

The same holds for the averaged tensors $\hat{A}_{ijkh}$ and $\hat{B}_{ijkh}$.

Therefore, only five independent components of the tensors A and B should be calculated:

$$\begin{gathered} A_{1111}, A_{1122}, A_{2233}, A_{1212}, A_{2323};\\ B_{1111}(t), B_{1122}(t), B_{2233}(t), B_{1212}(t), B_{2323}(t). \end{gathered}$$

So the resulting creep medium is transversely isotropic.

4. LAYERED MEDIA MODELING

For verification of the described method, the results of numerical calculations for the homogenized medium model are compared with the results of computations for the model, that does not use the homogenization. The medium consists of layers of two different materials. Different load types are investigated. The cube sample is examined, that has size $1 \times 1 \times 1$ (all quantities are dimensionless). This cube contains 10 layers of the material 1 and 10 layers of the material 2 having the same size (Fig. 2). The problem is solved in dimensionless form. The following materials properties are used:

$$\begin{gathered} E_1 = 2000; \nu_1 = 0.4; \delta_{c1} = 1; \delta_{v1} = 1; \alpha_1 = 0.2;\\ E_2 = 1000; \nu_2 = 0.2; \delta_{c2} = 1; \delta_{v2} = 1; \alpha_2 = 0.78. \end{gathered}$$

Note that since α is positive, the creep and relaxation kernels of each layer do not have singularities. Influence of the singularity on the computation accuracy is not considered in this study and should be examined separately, as well as the influence of the integration method in a neighborhood of the singularity.

Three problem statements are considered. In the first one, the equilibrium equations are solved for the medium with the directly modeled layers. The finite elements mesh

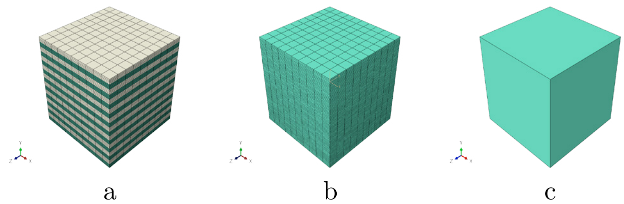

Fig. 2. Finite element mesh in the cases of the direct layered medium model (a), the homogenized model with the same mesh (b), and the homogenized model consisting of only one element (c)

consists of 8000 hexahedral elements: 10 × 10 in the horizontal plane and 4 elements per each layer in the vertical direction (see Fig. 2a). In the second one, the homogenized model of the medium is used on the same mesh (see Fig. 2b). In the third problem statement, the homogenized model of the medium is used again, but the mesh consists of only one finite element. (Fig. 2c). The problems examined below lead to the homogeneous stress-strain state in the averaged material model. Therefore, the results obtained in the second and in the third cases must coincide. The 4 orders mesh size reduction demonstrates the practical efficiency of the method of homogenization.

The computation is performed on the time interval $t = 0 \ldots T$ with the variable adaptive integration step. The first step is equal to $\Delta t = 0.01$. Homogenized tensors with 100 steps per period are used at the results presented below for the homogenized model. All the problems are solved in a geometrically linear formulation.

The numerical experiments are performed for the following types of external loads: the uniaxial stress orthogonal to the layers, the uniaxial stress along the layers, the shear parallel to the layers, the shear orthogonal to the layers, the full compression. For all this typical cases, the difference in strains is less than 5% for the exact model as compared to the homogenized one for the typical process times. More details are given below on the results of calculations in the cases of the uniaxial stress orthogonal to the layers, the uniaxial stress along the layers, and the shear parallel to the layers.

4.1 Uniaxial stress orthogonal to the layers

The bottom face of the cube is rigidly fixed (the displacements are set to zero). The stretching stress $\sigma_{22} = 10$ is applied to the top face. The case of uniaxial tension corresponds to the stretching-compression experiments for material samples. Each layer tends to change in cross-sectional size in different degree due to the difference in the Poisson's ratio. As a result, the solution of this problem for the layered medium becomes essentially three-dimensional. The following problem is considered with the purpose to obtain simpler solutions. The additional condition of zero horizontal displacements $u_1 = u_3 = 0$ is imposed on all points of the medium. This corresponds to the condition $\varepsilon_{11} = \varepsilon_{33} = 0$ for the strain tensor components. The solution of this problem problem depends only on one spatial coordinate y. Thus, the conditions correspond to the uniaxial medium deformation. The calculated strain tensor component $\varepsilon_{22}(t)$ is shown in Fig. 3 for the described above three problem statements. Here and below in the paper, the calculated for the layered model strain tensor is then averaged over the layers. Therefore, the resulting strain tensor component $\varepsilon_{22}(t)$ shown in the figure is constant in space for the all three cases. There is an exact match for different meshes for the homogenized model. The difference between the homogenized and non-homogenized models is 3.2% at the moment of time $t = T$, where $T = 3$.

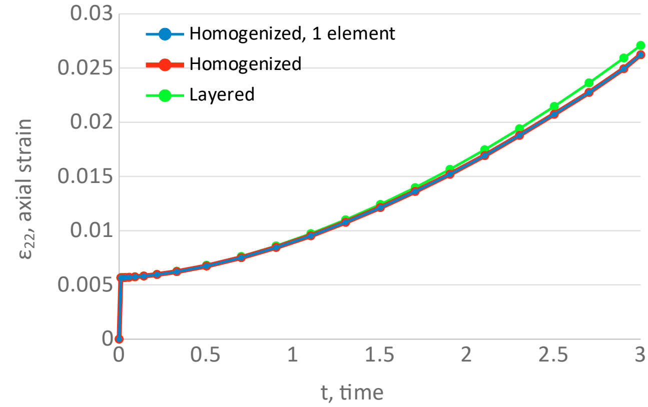

Fig. 3. The strain tensor component $\varepsilon_{22}(t)$ for three problem statements

4.2 Uniaxial stress along the layers

The rear face of the cube is rigidly fixed (the displacements are set to zero). The stretching stress $\sigma_{11} = 10$ is applied to the front face. As in the previous example, the additional condition of zero horizontal displacements orthogonal to the load direction $u_2 = u_3 = 0$ is imposed on the all medium points. This condition corresponds to the strain tensor components $\varepsilon_{22} = \varepsilon_{33} = 0$. It allows to simulate the uniaxial deformation. The resulting strain tensor component $\varepsilon_{11}(t)$ is shown in Fig. 4 for the three problem statements described above. There is an exact match for different meshes for the homogenized model. The difference between the homogenized and non-homogenized models is 1.2% at the moment of time $t = T$.

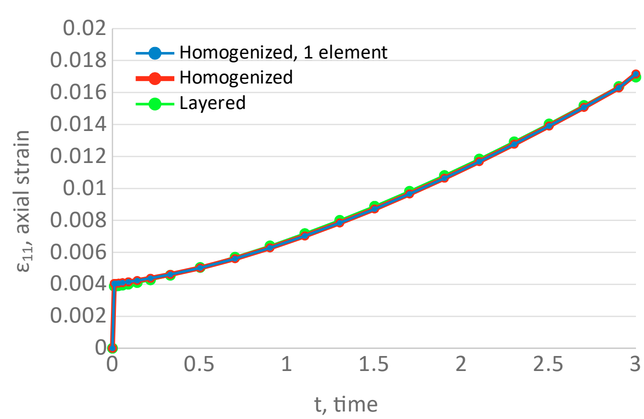

Fig. 4. The strain tensor component $\varepsilon_{11}(t)$ for three problem statements

4.3 Shear parallel to the layers

The bottom face of the cube is rigidly fixed (the displacements are set to zero). The tangent stress $\sigma_{12} = 10$ is ap-

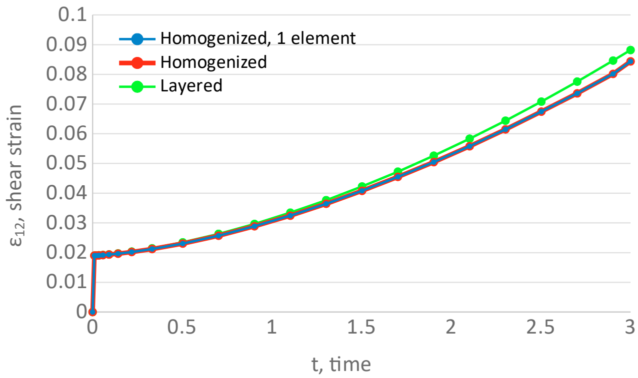

Fig. 5. The strain tensor component $\varepsilon_{12}(t)$ for three problem statements

plied to the top face. The additional condition $u_2 = u_3 = 0$ is imposed on the all medium points. This condition corresponds to the strain tensor components $\varepsilon_{22} = \varepsilon_{33} = 0$.

The strain tensor component $\varepsilon_{12}(t)$ is shown in Fig. 5. As in the previous examples, there is an exact match for different meshes for the homogenized model. The difference between the homogenized and non-homogenized models is 4.3% at the moment of time $t = T$. The presented in the figure deformation is a result of averaging on the layers. The layered medium experiences a form distortion, which occurs due to the difference in properties of layers.

4.4 Comparison with naive homogenization

In some technical applications the naive homogenization is used. In this case the effective tensors and creep kernels are specified as the simple arithmetic means of the corresponding tensors and kernels of the non-homogenized layers. The results obtained in this case are compared with the exact homogenization results. This comparison is shown in Fig. 6. The shear load along the layers is applied to the sample. The resulting shear strain is shown. It is shown that the accuracy of the naive homogenization becomes inappropriate from the very beginning of the considered load time interval.

5. CONCLUSION

In the present paper the method of homogenization of Shamaev and Shumilova (2016) is used to create the homogenized model of the layered creep media. The time steps method of Konstantinova and Chernopazov (2004) is used for simulation of different loads applied to the cubic sample on considerable time intervals. It is shown that the homogenized model is sufficiently accurate. The difference is less than 5% as compared with the results obtained with the direct computations when non-homogenized model is utilized. This tolerance holds for rather long time. The sample can be deformed by 10-15% by the end of the considered time period.

An inverse number of layers serves as a small parameter in the homogenization theory. The proximity of the homogenized solution to the original one is estimated in terms of this small parameter. Thus, it is critical to a homogenization theory application, that the number of layers is sufficiently large. Here, it is shown that 10 layers

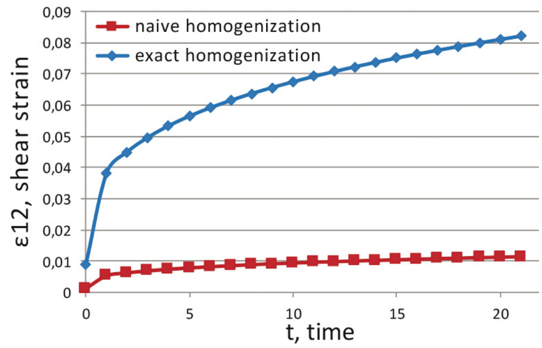

Fig. 6. Naive and exact homogenizations

is sufficient for obtaining the described accuracy of the homogenized model. Application of homogenized model allows to decrease the number of finite elements by 4 orders. This leads to the same decrease of the computation time and required computer memory.

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